# TDA - fall 2025

Homology (finally!)

# Recap

- first HW is due today
  ↳ upload pdf to Canvas
- second HW - posted after class

# Last time: Chain complexes

complex

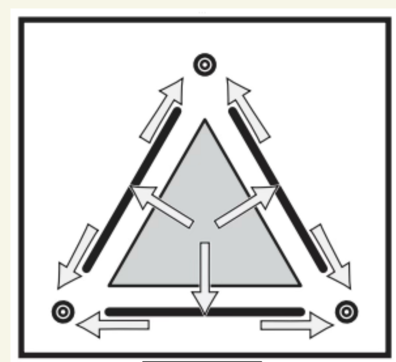
A p-chain: formal sum of p-simplices in K:

$$\alpha = \sum a_i \sigma_i \qquad a_i \in \mathbb{Z}_2$$

$$\cdots \to C_{p+1}(K) \xrightarrow{\partial_{p+1}} \overbrace{C_p(K)}^{p^{\text{th}}\text{ chain group}} \xrightarrow{\partial_p} C_{p-1}(K) \to \cdots$$

Boundary maps $\partial_p$: for any simplex $\sigma \in C_p$,

$$\partial_p(\sigma) = \sum \text{p-1 simplices share p vertices}$$



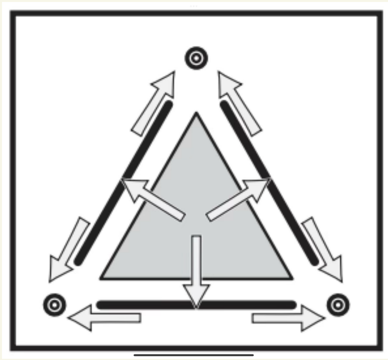
Cycles $Z_p \subseteq C_p$: $\partial_p(x) = 0$

ker $\partial_p$

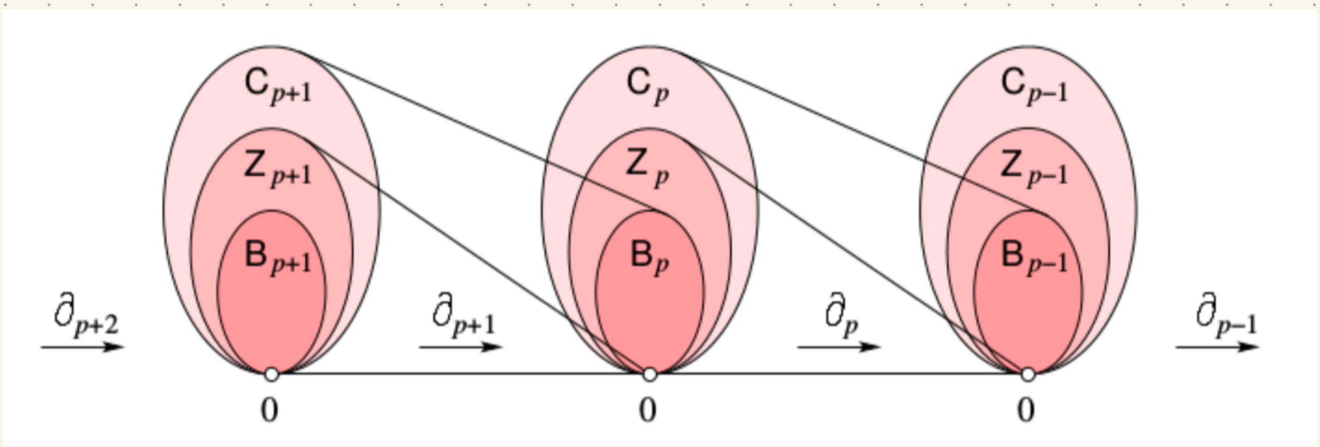
Boundaries $B_p \subseteq C_p$: elements are hit by $\partial_{p+1}$ = im $\partial_{p+1}$

Note: Since $\partial_p \partial_{p+1}(\alpha) = 0 \ \forall \alpha \in C_{p+1}(K)$

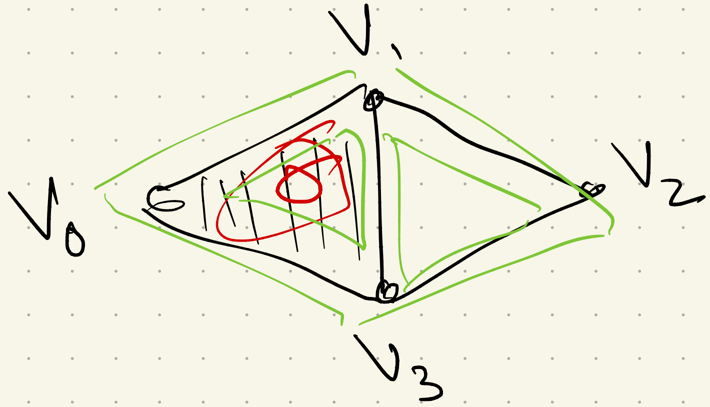
$\Rightarrow$ every p-boundary is also a p-cycle



So we get:



## Example:

$K =$ 

$$C_1 \supseteq B_1$$

Generators of $B_1(K)$?

look at $C_2$: $\sigma = [v_0 v_1 v_3]$

$\rightarrow \partial_2(\sigma) = \{v_0 v_1 + v_1 v_3 + v_0 v_3\}$

Generators of $Z_1(K)$?

$\ker \partial_1$ sets

$v_1 v_2 + v_2 v_3 + v_1 v_3 = \alpha_1$

$v_0 v_1 + v_0 v_3 + v_2 v_3 = \alpha_2$

$v_1 v_2 + v_2 v_3 + v_3 v_0 + v_0 v_1 = \alpha_3$

these 2 generate any other cycle

$\alpha_3 = \alpha_1 + \alpha_2$

# Quotient space

Take a vector space $V$ over field $F$, and $W \subset V$ a subspace.

Define $\sim$ on $V$ by $x \sim y$ iff $x - y \in W$.

**Equivalence class of $x$:**

$$[x] = x + W = \{x + w \mid w \in W\}$$

$$y \in [x] \Rightarrow x - y \in W$$

Then, quotient space $V/W$ is $\{[x] \mid x \in V\}$.

**Fact:** $V/W$ is a vector space with

- Scalar multiplication

$$a[x] = [ax]$$

- Addition:

$$[x] + [y] = [x+y]$$

# Homology

The $p^{th}$ homology group is the quotient space:

$$H_p(K) := Z_p(K) \Big/ B_p(K)$$

Im $\partial_{p+1}$

ker $\partial_p$

Recall: $C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1}$

$$[\alpha] \in H_p(K) \qquad \alpha \in Z_p$$

$\boxed{\alpha \text{ is a cycle}}$

$$\Rightarrow \{\alpha + \beta \mid \beta \in B_p\}$$

$$= \{\alpha + \partial_{p+1}\gamma \mid \gamma \in C_{p+1}\}$$

We say $\alpha, \beta \in C_p(K)$ are homologous

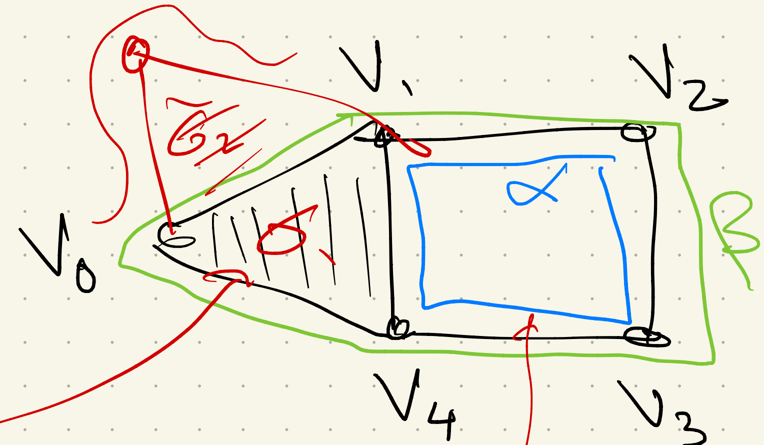
if $[\alpha] = [\beta]$ in $H_p(K)$

so: $\alpha = \beta + \partial\gamma$ for $\gamma \in C_{p+1}(K)$

cycle  cycle  boundary of higher dim chain

Time for an example ...

Can we find homologous 1-cycles?

$K =$ 


Consider: $\alpha = v_1v_2 + v_2v_3 + v_3v_4 + v_1v_4$

$\beta = v_1v_2 + v_2v_3 + v_3v_4 + v_0v_4 + v_0v_1$

If homologous, need a 2-chain $\gamma$ s.t.

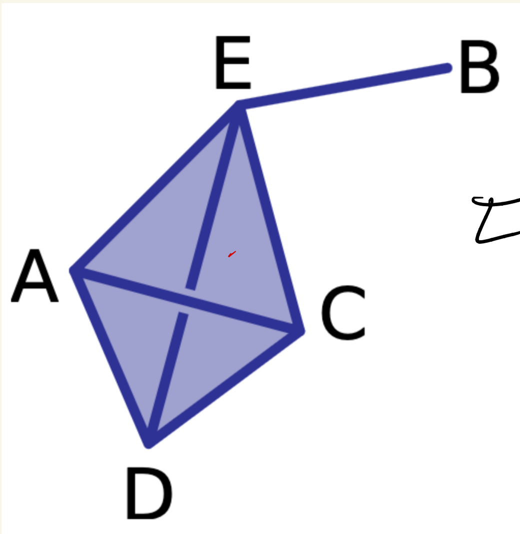
$\alpha = \beta + \partial_2 \gamma$.

$\lambda = \overline{v_0v_1 + v_1v_4 + v_0v_4} = [0]$

choice. could also use $\beta$

Here, $H_1(K) = \langle \underset{\sim}{\alpha, 0} \rangle = \langle \underset{\sim}{\alpha, \lambda} \rangle$

Another: What is $H_2(K)$?




no tetrahedron inside this time!

Well: $C_3(K) \xrightarrow{\partial_3} C_2(K) \xrightarrow{\partial_2} C_1(K)$

& $H_2(K) = \ker(\partial_2) / \mathrm{im}(\partial_3)$

$= Z_2 / B_2$

What is in $\mathrm{im}(\partial_3)$?

$\mathrm{Im}(\partial_3) = 0$ (nothing in $C_3$)

What about $\ker(\partial_2)$? $\gamma = acd + ace + cde + aed$

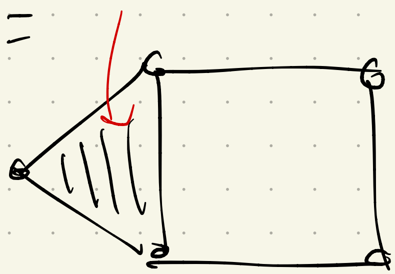
chain of $\triangle$'s, cancels under $\partial_2$

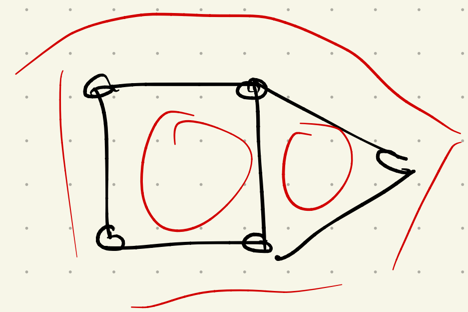
$Z_2 / \cancel{B_2}^{=0}$

So: $H_2(K) = \langle \gamma, 0 \rangle$

# Betti numbers

The $p^{th}$ Betti number is the rank of the p-dim homology: $\beta_p = \text{rank}(H_p)$

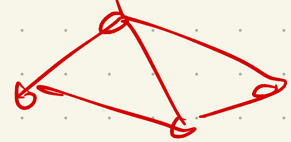
$K_1 =$ 

$K_2 =$ 

$\beta_1(K_1) = 1$

$\beta_1(K_2) = 2$

# Some common spaces

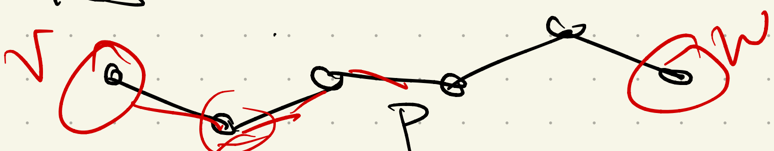


① Graphs: 1d simplicial spaces

$$C_2(G)=0 \xrightarrow{\partial_2} C_1(G) \xrightarrow{\partial_1} C_0(G) \xrightarrow{\partial_0} \phi$$

im $\partial_2 = 0$

- $\partial_0 = 0$, so every vertex is a 0-~~cycle~~ chain & cycle
- $B_0$: boundaries of 1-chains (=paths)




$\partial(p)$ = endpoints

v, w

- So $H_0(G)$ = connected components of G
- For $H_1$: no 2-cells! $\Rightarrow$ $B_1 = 0$

What is $Z_1$? any cycle

basis for $H_1$: minimum cycle basis

## ② Surfaces:

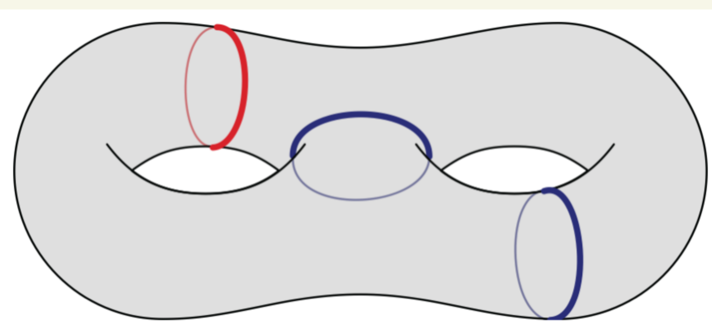
$$\underbrace{C_3 = \emptyset}_{} \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \emptyset$$

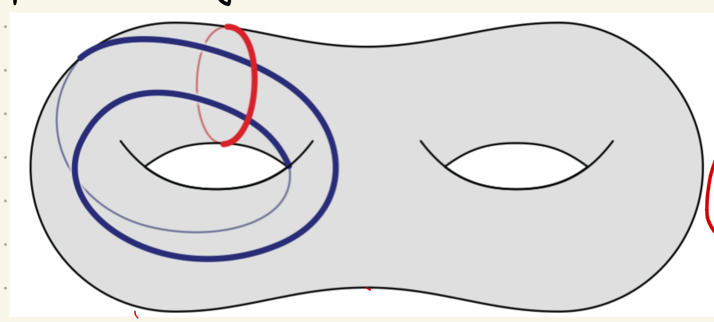
$\Rightarrow \operatorname{rank}(H_0) = 1$

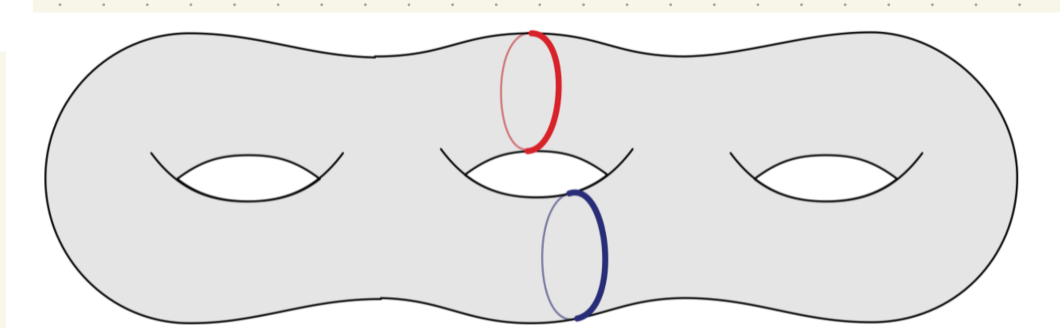
- $H_0$: same as graphs
- $H_1 = Z_1 / B_1 = \ker(\partial_1) / \operatorname{im}(\partial_2)$

  $Z_1$: Still unions of cycles

  $B_1$: differ by $\partial$(some $\cup$ of $\Delta$'s)



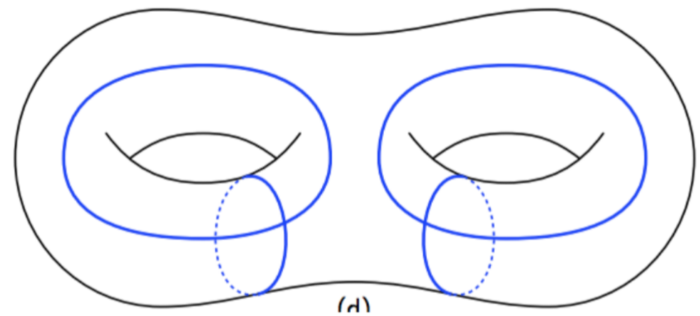




Surfaces (cont):

In the end: non-zero for $H_0$, $H_1$, & $H_2$:

$$\dim H_k(S_g) = \begin{cases} 1 & : \quad k = 0 \\ 2g & : \quad k = 1 \\ 1 & : \quad k = 2 \\ 0 & : \quad k > 2 \end{cases}.$$



Erickson-Whittlesey 2005

$H_2$: the only 2-cycle is the union of all $\Delta$'s

$H_1$: 2g cycles per handle

# Computing homology groups

To compute Betti number:

$$\beta_p = \dim(H_p(K))$$

Well, for any linear transformation $f: U \to V$,

$$\dim(U) = \dim(\ker f) + \dim(\operatorname{im} f)$$

set $f = \partial_p$:

$\partial_p : C_p \xrightarrow{\partial_p} C_{p-1}$

$$\dim(C_p) = \dim(\ker \partial_p) + \dim(\operatorname{im} \partial_p) = Z_p + B_{p-1}$$

Also, for a quotient space $V/W$,

$$\dim(V/W) = \dim(V) - \dim(W)$$

$$\Longrightarrow \beta_p = \dim(Z) - \dim(B)$$

So: computing!

Back to boundary matrices:

$$\partial_p \circ \underbrace{\alpha}_{\substack{\text{~~p-simplex~~} \\ \text{p-chain}}} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n_p} \\ b_{2,1} & & & b_{2,n_p} \\ \vdots & & & \\ b_{n_{(p-1)},1} & \cdots & & b_{n_{p-1},n_p} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_{n_p} \end{bmatrix} = \text{p-1 chain}$$

Rows are a basis for $C_{p-1}$

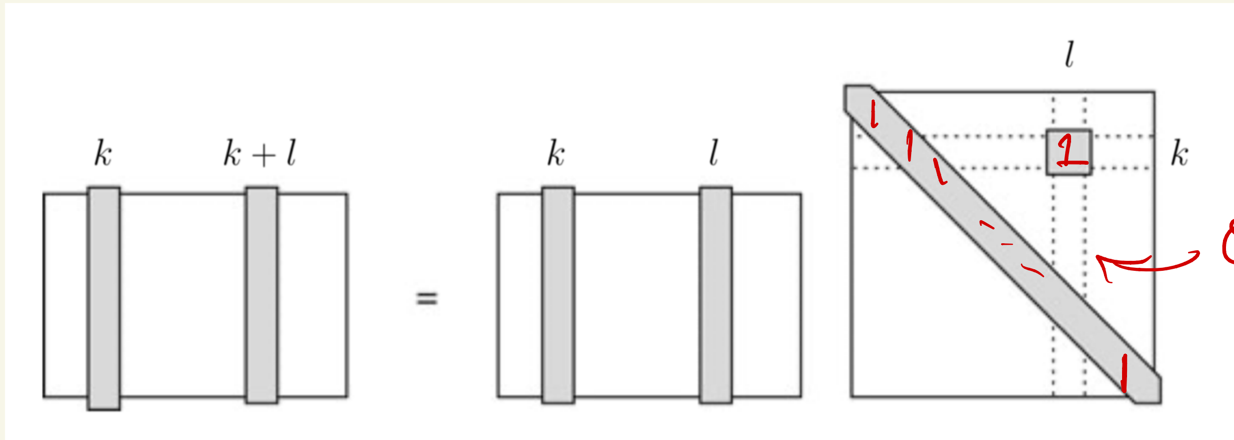
Columns are a basis for $C_p$

How to find rank?

# Operations on matrices

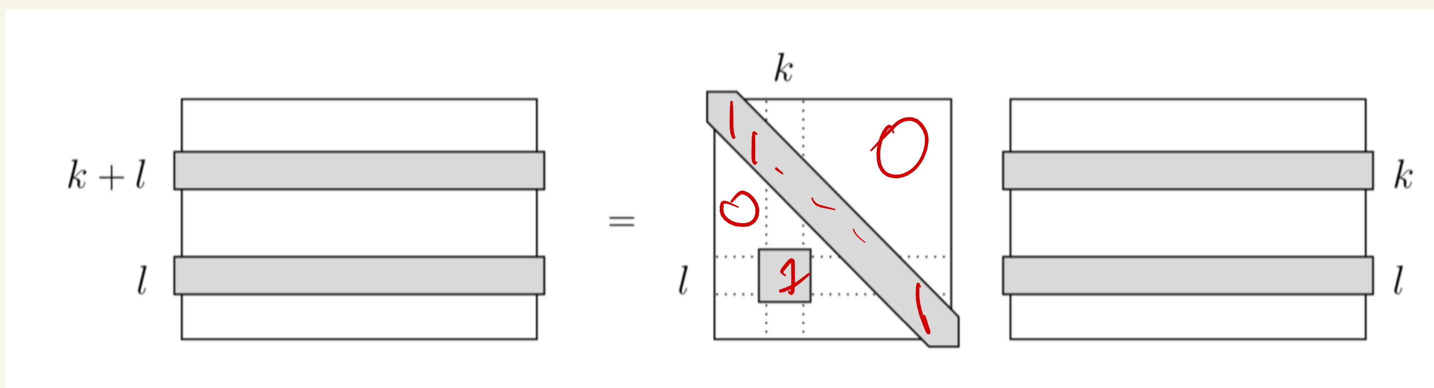
Simplify to Smith-Normal form. How?

Add columns




0 all other spots

Add rows

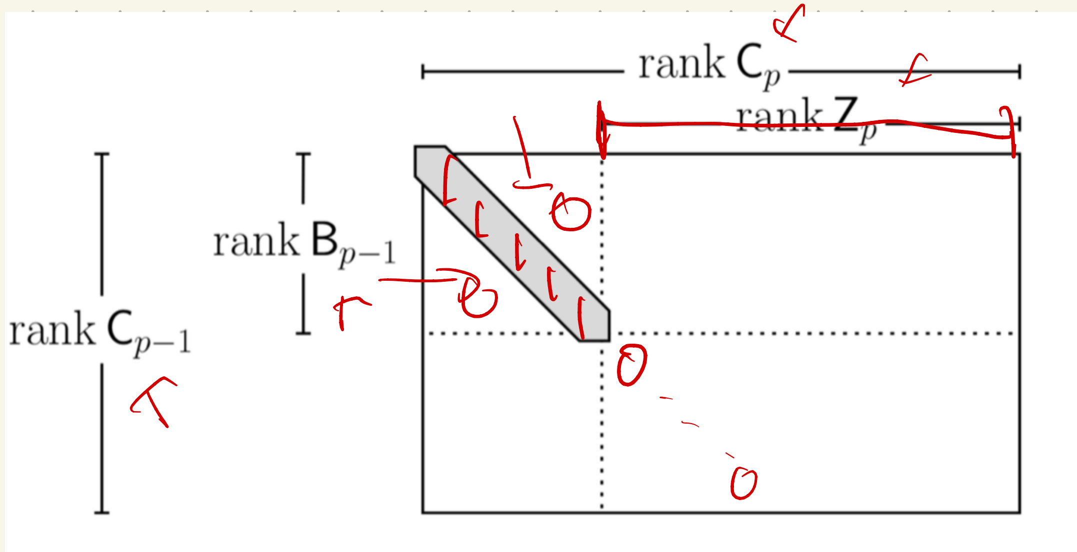



(& exchange rows/columns with 0 on diagonal)

Goal: Move 1's to diagonal

Smith-Normal form: $N_p = U_{p-1}\,\partial_p\,V_p$

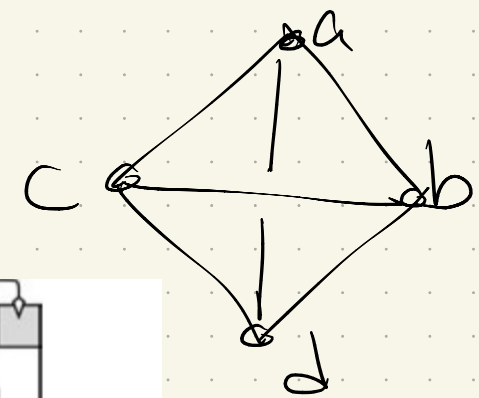
$N_p =$



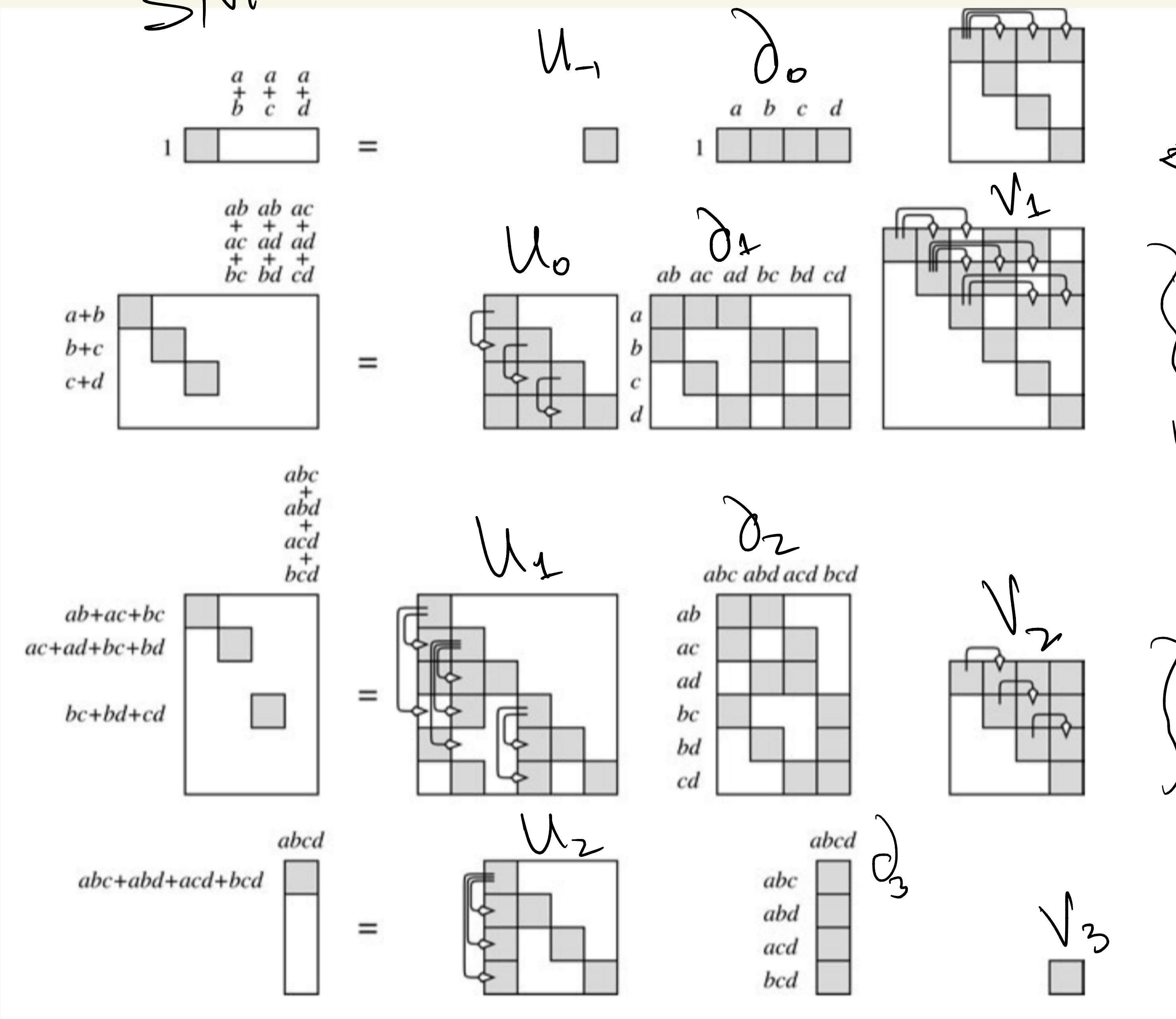

then $\beta_p = \text{rank}(Z_p) - \underbrace{\text{rank}(B_p)}_{?}$

$N_{p+1} =$ I $\leftarrow$ rank $B_p$

# An example: solid tetrahedron




SNF




rank $Z_0$ =

rank $B_0$ =

rank $Z_1$ =

rank $B_1$ =

rank $Z_2$ =

rank $B_2$ =

Next time:

Simplicial Maps & induced homology

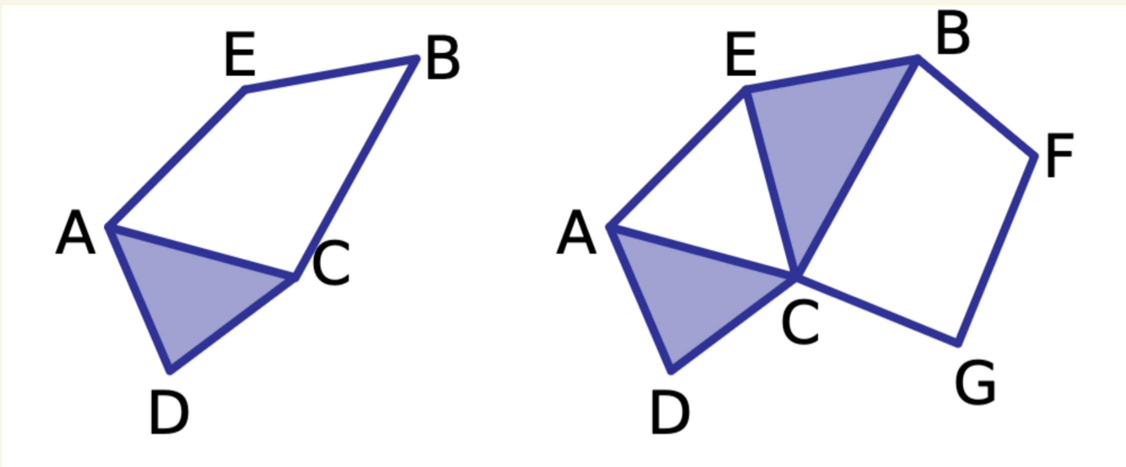
Diagram Chasing

& hopefully persistent homology

Recall: A simplicial map between abstract simplicial complexes $f: K \to L$ is induced by a map on vertices $V(K) \to V(L)$

Inclusion maps: $i: K \to L$, $K \subseteq L$

$i(\sigma) = \sigma$



# Passing to chain complexes

Any $f: K \to L$ naturally extends to a map on chain complexes:

$$f_\#: C_p(K) \longrightarrow C_p(L)$$

Results in a diagram:

$$\cdots \to C_{p+1}(K) \longrightarrow C_p(K) \longrightarrow C_{p-1}(K) \to \cdots$$

$$\cdots \to C_{p+1}(L) \longrightarrow C_p(L) \longrightarrow C_{p-1}(L) \to \cdots$$

Claim: $f_\# \partial^K = \partial^L \circ f_\#$

Why?

$$
\begin{tikzcd}
C_p(K) \arrow[r, "\partial_p^K"] \arrow[d, "f_\#"'] & C_{p-1}(K) \arrow[d, "f_\#"] \\
C_p(L) \arrow[r, "\partial_p^L"] & C_{p-1}(L)
\end{tikzcd}
$$

Consider a $\sigma \in C_p(K)$.

Claim: $f_\#(\text{cycle in } K) = \text{cycle in } L$

$f_\#(\text{boundary in } K) = \text{boundary in } L$

Why?

Because it commutes!

Consider a cycle:

$\partial_p^K(\alpha) = 0$

$$\begin{tikzcd}
C_p(K) \arrow[r, "\partial_p^K"] \arrow[d, "f_\#"'] & C_{p-1}(K) \arrow[d, "f_\#"] \\
C_p(L) \arrow[r, "\partial_p^L"] & C_{p-1}(L)
\end{tikzcd}$$

This induces a map on homology:

$$f_* : H_p(K) \longrightarrow H_p(L)$$

$$[\alpha] \longmapsto [f_\#(\alpha)]$$