TDA - fall 2025

Filtrations
& persistence

Last time:

- Homology $H_p(K)$: generated by p-cycles which are not homologous; differ by a $(p+1)$ boundary
- Induced Homology : $i: L \hookrightarrow K$
  then have $i_*: H_p(L) \longrightarrow H_p(K)$
- Relative Homology : $L \subseteq K$
  $H_p(K, L)$: homological features in K/L

Filtrations

A filtration $F(K)$ of a simplicial complex K is a nested sequence of subcomplexes:

$$F: \emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$$

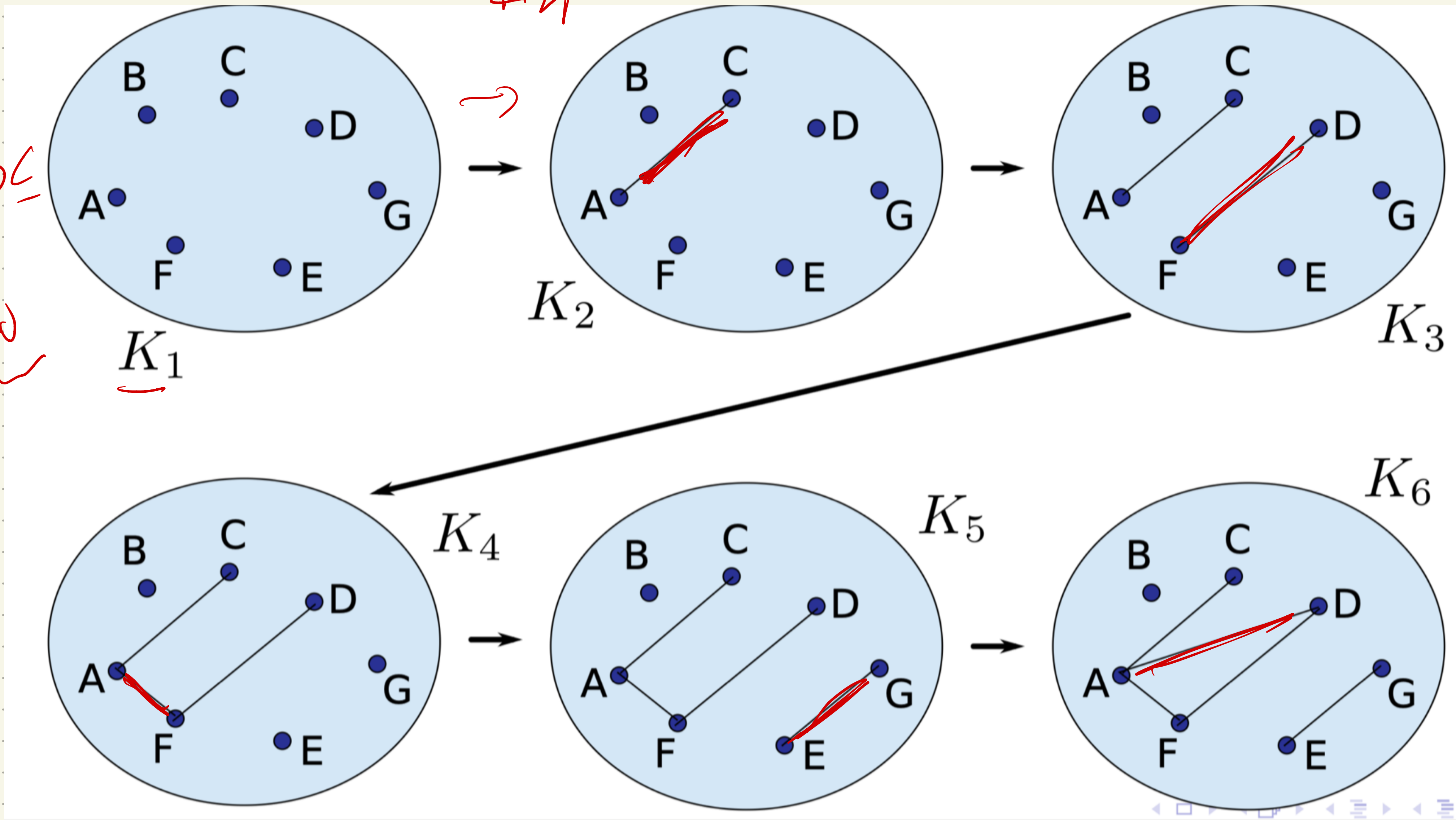
Called simplex-wise if K_i / K_{i-1} is empty or a single simplex.

Example 1:

K_2/K_1

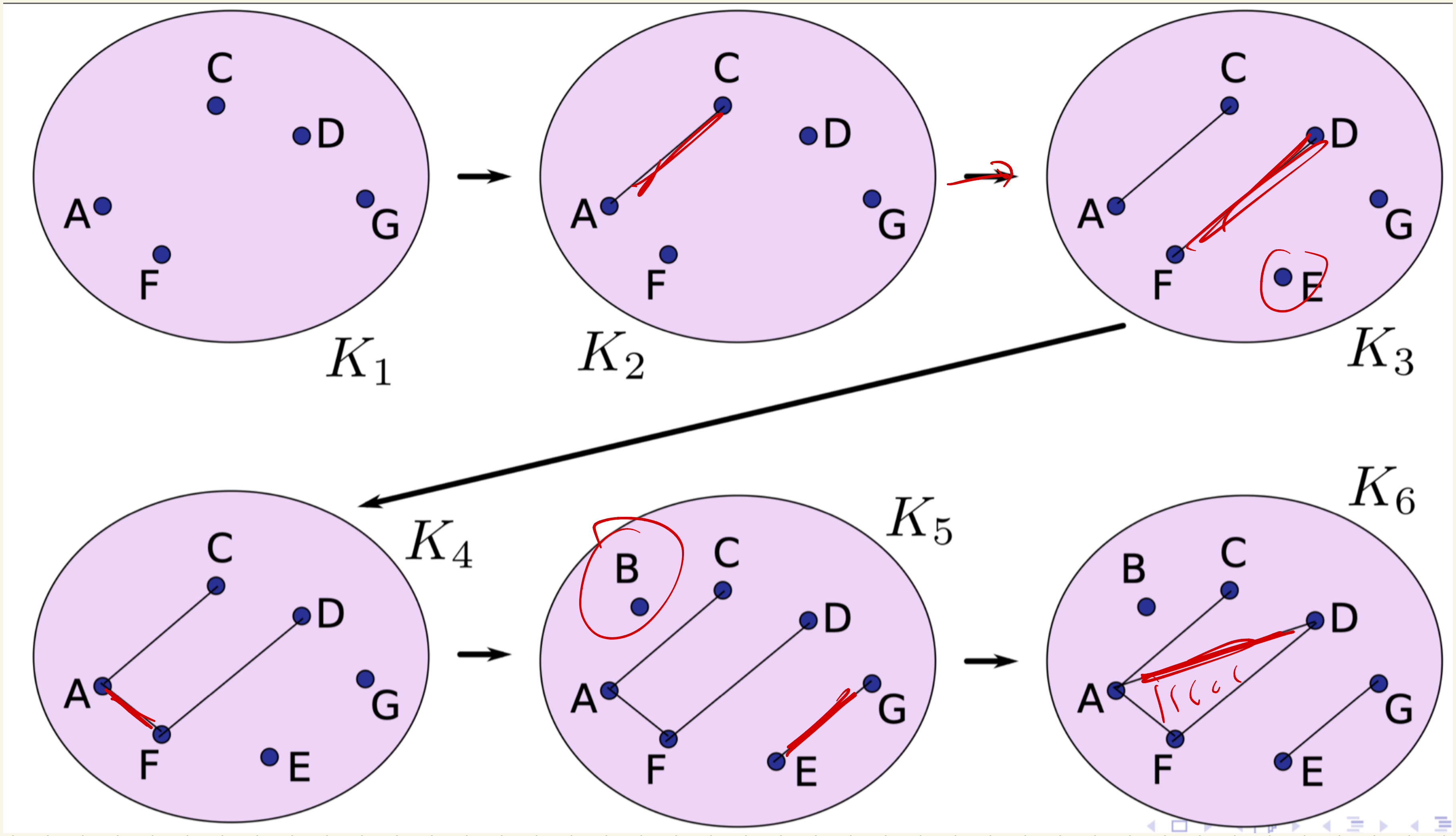
$\phi \subseteq$

K_1



Example 2

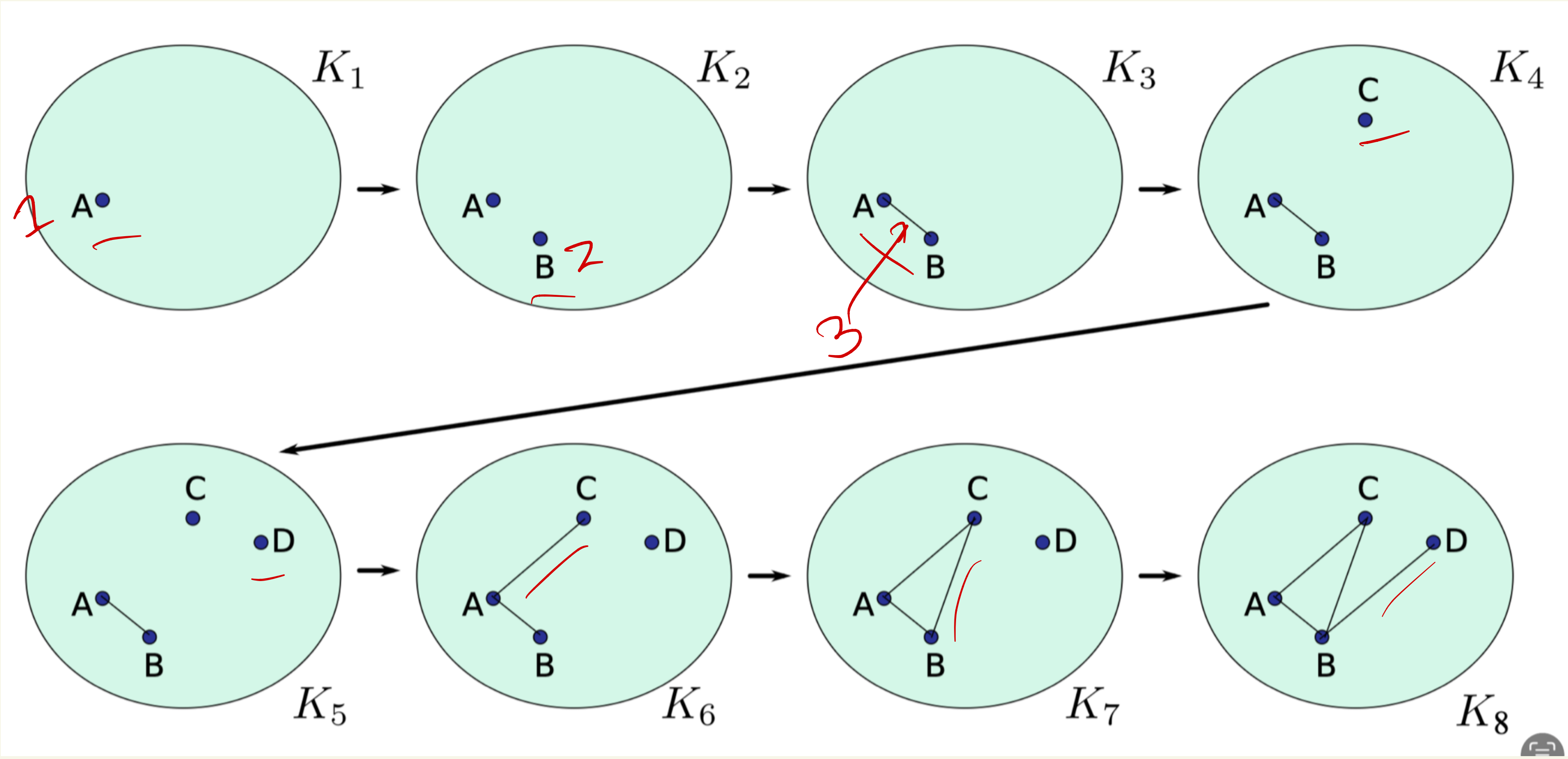
$\overline{AC}$ $\qquad$ $\overline{FD}$ & E



not simplexwise

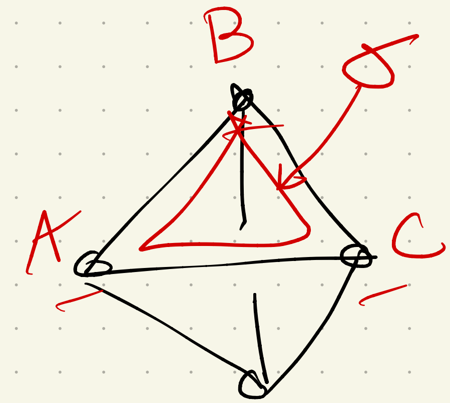
Example 3

simplex-wise



Functions & filtrations

Consider $f : K \to \mathbb{R}$.



f is monotone if $\forall \tau \le \sigma$,

$$f(\tau) \le f(\sigma)$$

$f(\sigma) = 10$

$$\left.\begin{array}{l} f(AB) \\ f(BC) \\ f(AC) \end{array}\right\} \le 10$$

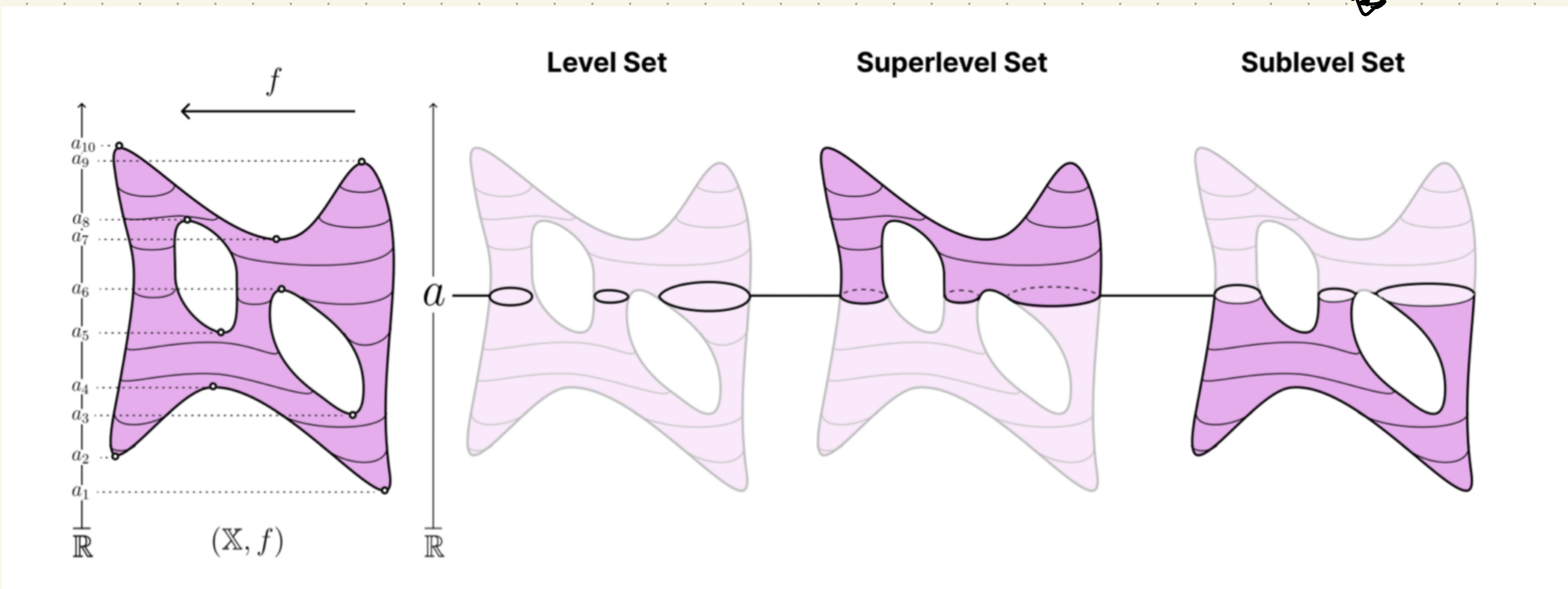
Fix $a_0 \le a_1 \le \cdots \le a_n$

& let $K_i = f^{-1}((-\infty, a_i])$.

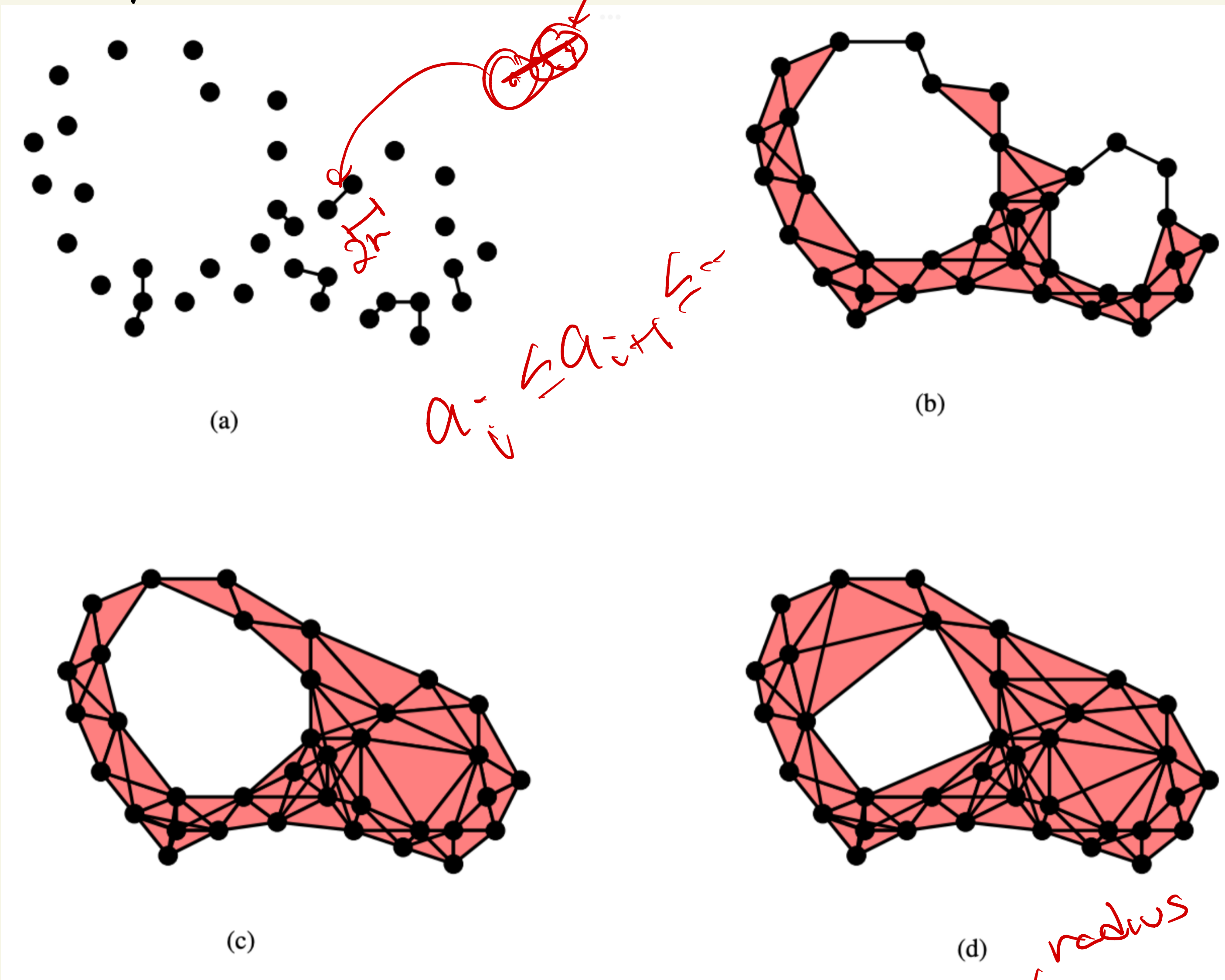
The sublevel set filtration induced by f is defined to be

$$\emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$$

Note: We saw natural extensions to non-simplicial setting, assuming "nice" functions + spaces.



Example: Čech complex



What is f?

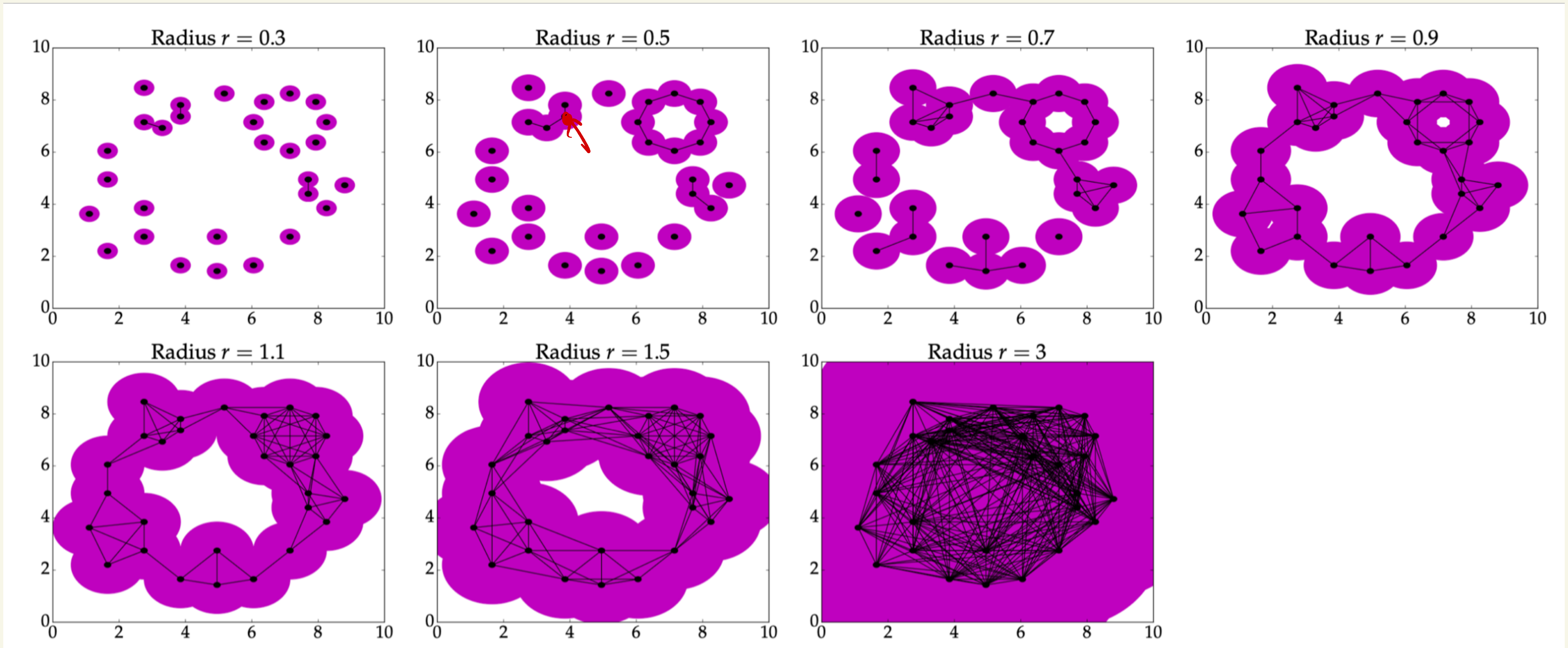
Consider $\sigma = [v_0 \cdots v_k]$

radius

$$f(\sigma) = \min\{a \mid B(a/2, v_0) \cap B(a/2, v_1) \cap \cdots \cap B(a/2, v_k) \neq \emptyset\}$$

vertex

Example: Rips filtration

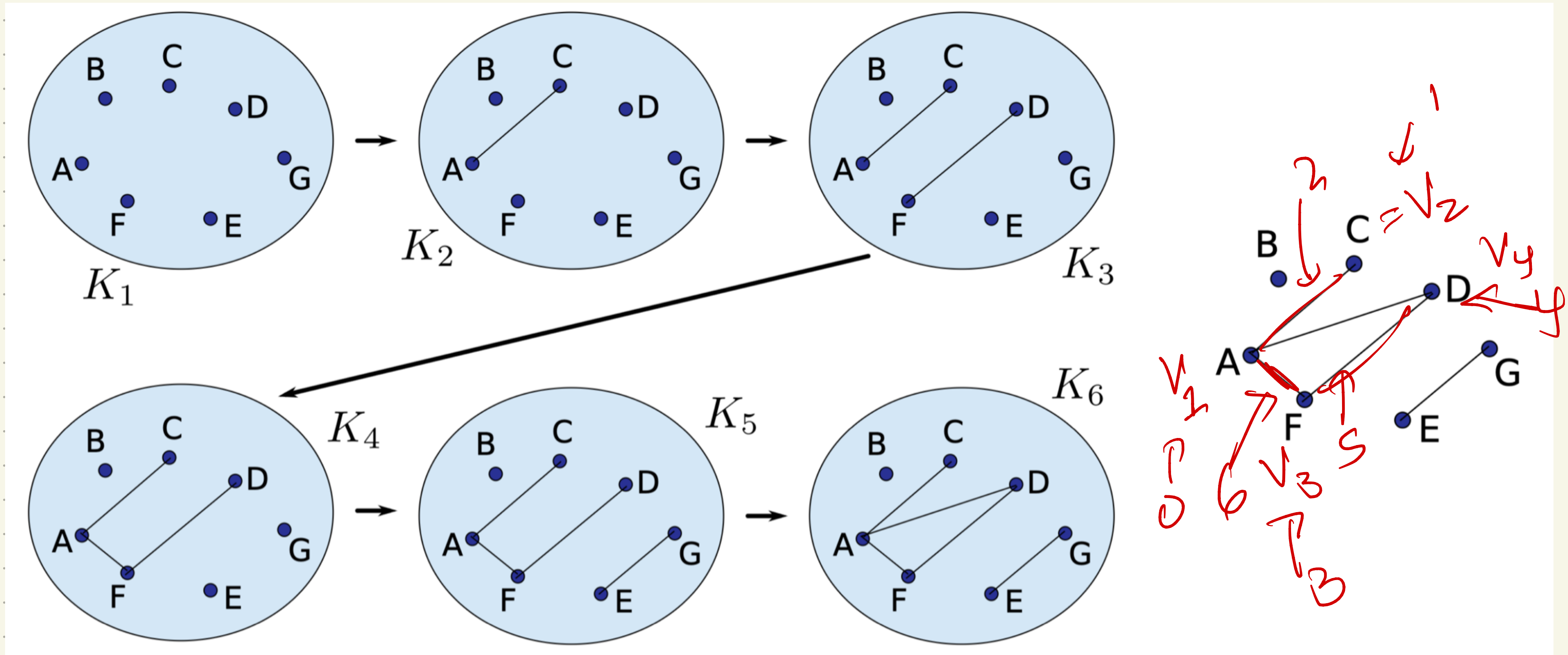


What is f?

$\sigma = [v_0 \dots v_k]$

$$f(\sigma) = \min \{ a \mid \forall i,j \in [0 \dots k],\ B(a/2, v_i) \cap B(a/2, v_j) \neq \emptyset \}$$

Common way to build f: vertex function



Lower & upper stars

Fix a total order on vertices:

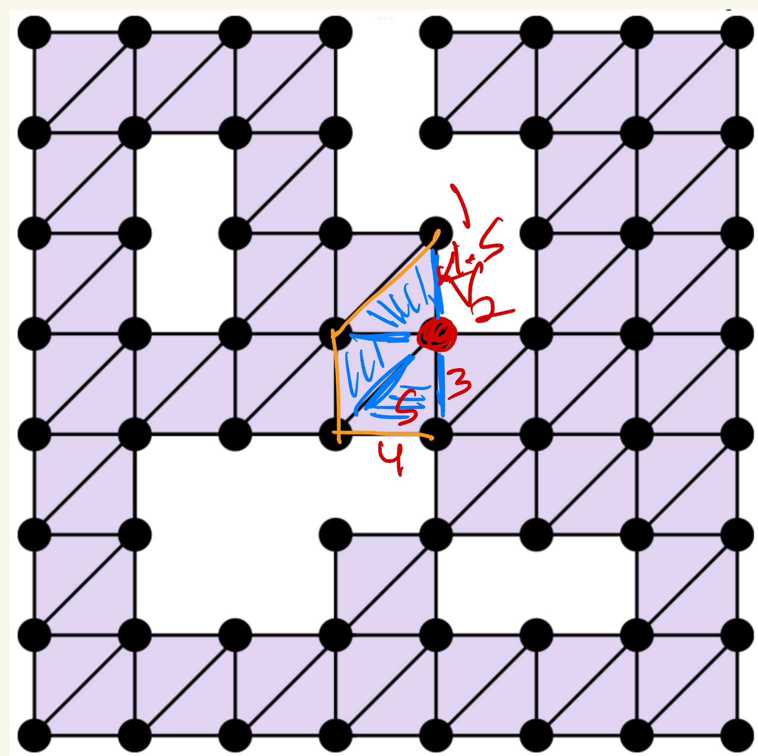
$$f(v_1) \le f(v_2) \le \cdots \le f(v_n)$$

Lower star of v, $Lst(v)$, is the set of simplices in $st(v)$ with lower function value.

Recall:

star, link, & closures:

$$\text{Link}(v) = \overline{St}(v) - St(v)$$



st(v)

link(v)

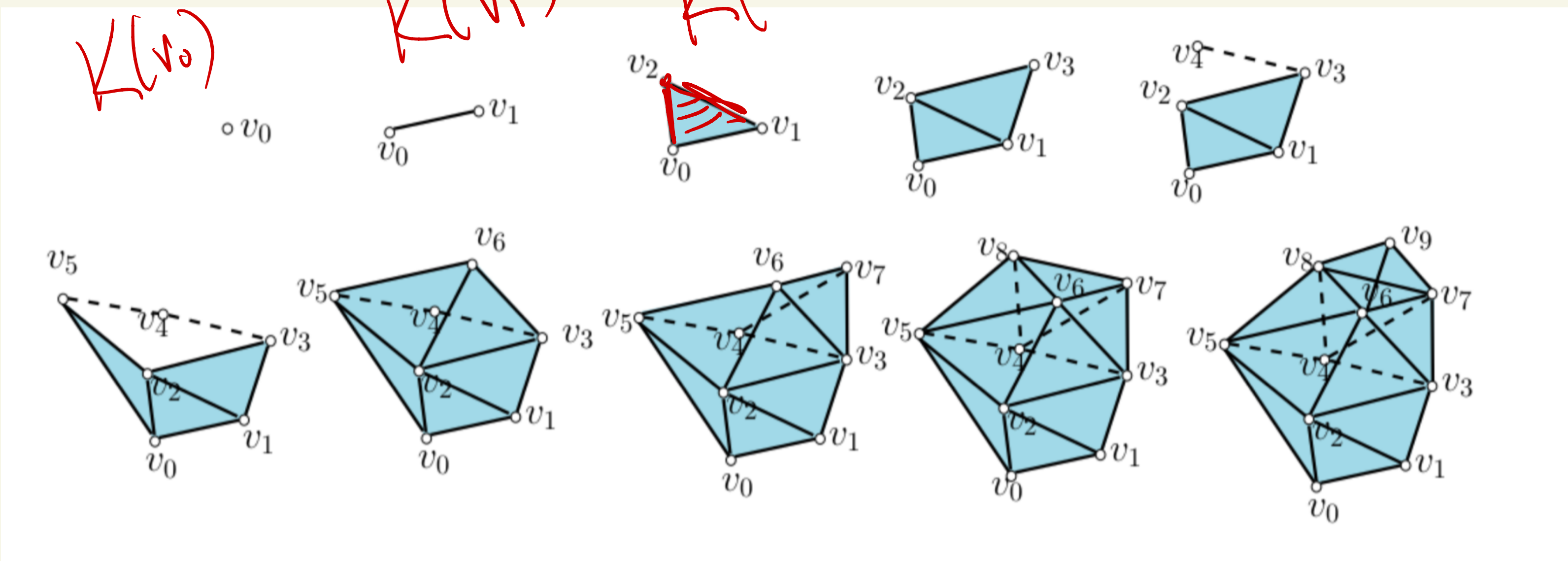
$f(v) = 2$

Lower star filtration:

$$\emptyset = K_{f(v_0)} \subseteq K_{f(v_1)} \subseteq \cdots \subseteq K_{f(v_n)} = K$$

where $K_{f(v_i)}$ is all simplices spanned by $v_1, \dots, v_i$

$K(v_0)$ $K(v_1)$ $K(v_2)$



Note: simplex-wise? NO

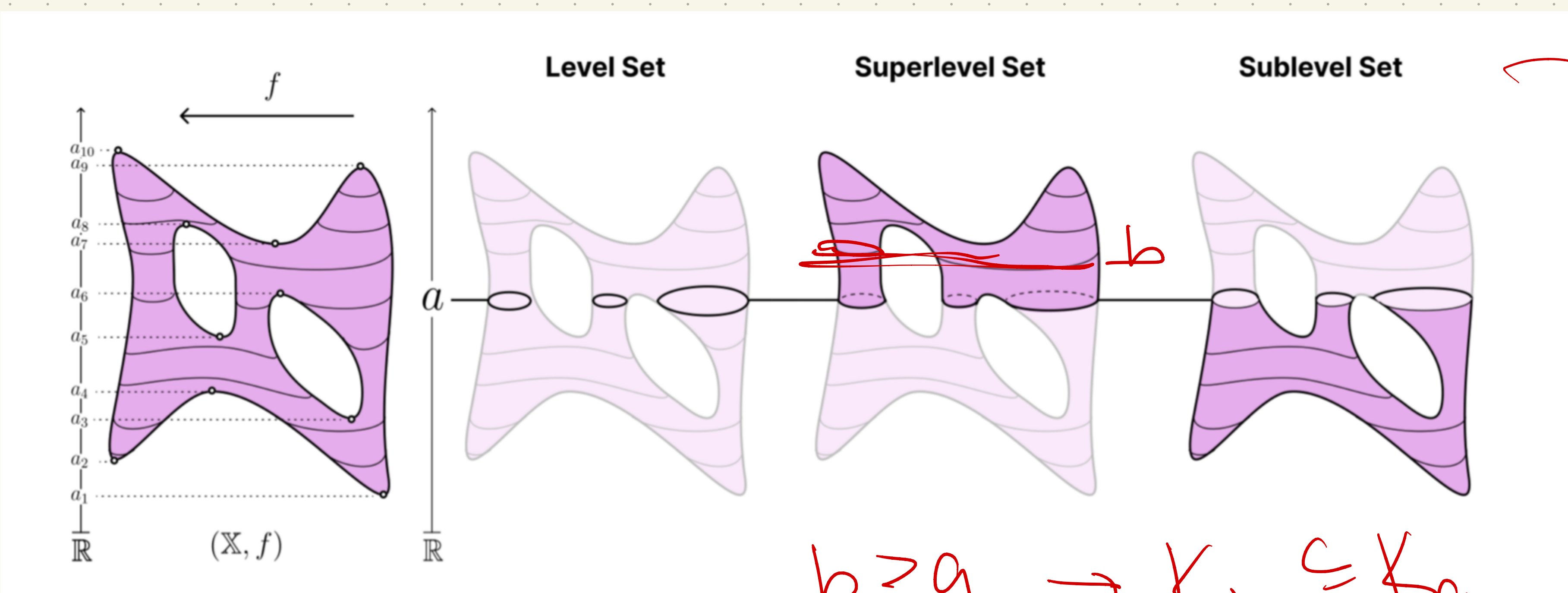
What is $K_{f(v_i)} / K_{f(v_{i-1})}$? lower star of v_i

lower link of v_i in K_{i-1}

Upper star filtration

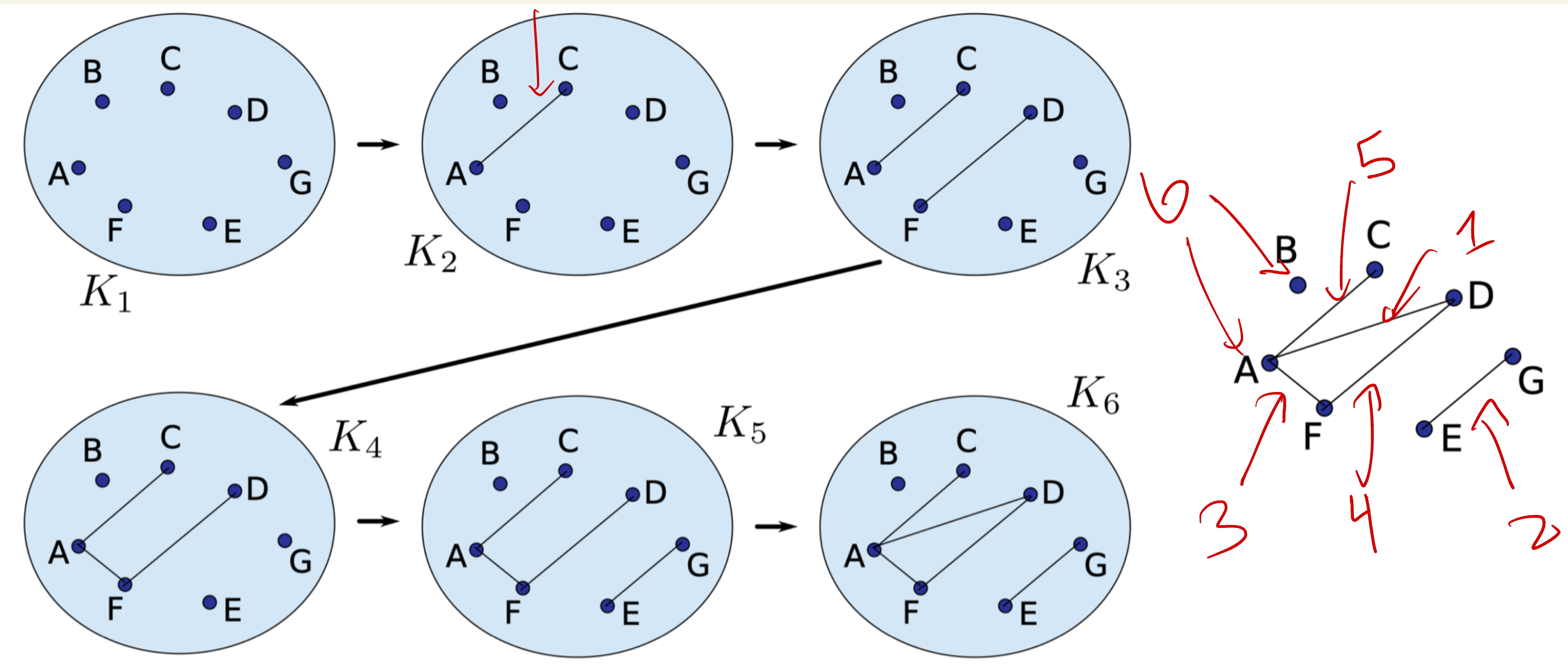
Same setup with vertex order, but set $K^{f(v_i)}$ = all simplices spanned by $v_i, v_{i+1}, \ldots, v_n$

then $\phi = K^{f(v_{n+1})} \subseteq K^{f(v_n)} \subseteq \cdots \subseteq K^{f(v_1)} = K$



$b > a \rightarrow K_b \subseteq K_a$

Try it: ~~vertex~~ ordering for superlevel set filtration:



Induced maps on homology

Each $K_i \hookrightarrow K_{i+1}$, so we get induced maps $H_p(K_i) \to H_p(K_{i+1})$.

Homology module (simplicial case):

$H_p(\mathcal{F}(K))$:

$$\begin{tikzcd} \emptyset = H_p(K_0) \arrow[r] & \underset{i}{H_p(K_1)} \arrow[r] \arrow[rr, bend left, "\text{inclusion}"] & \cdots \arrow[r] & \overset{j}{H_p(K_n)} = H_p(K) \end{tikzcd}$$

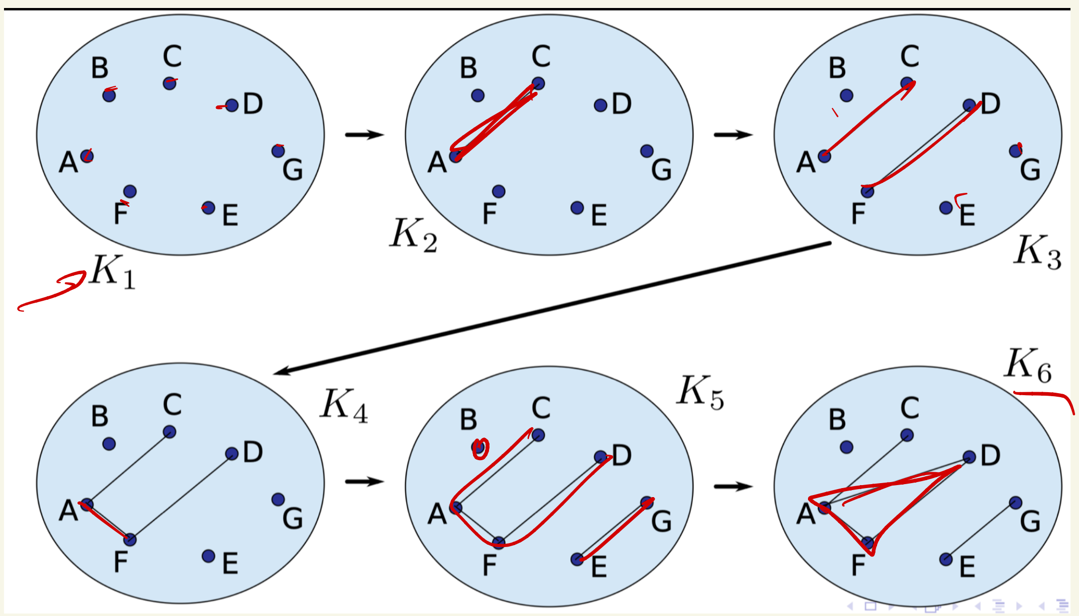
$i, j \to f_p^{i,j} : H_p(K_i) \to H_p(K_j)$

What do these capture?

Simple first case: Both curves

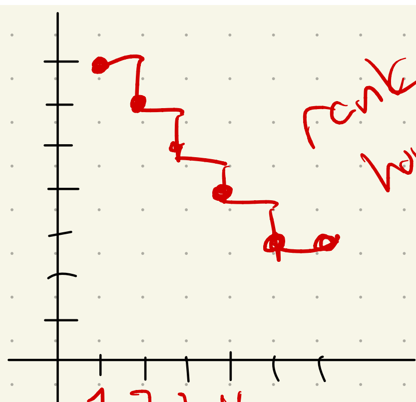
$$\beta_p(\mathcal{F}(K)) : \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$i \longmapsto \beta_p(K_i) = \text{rank}(H_p(K_i))$$



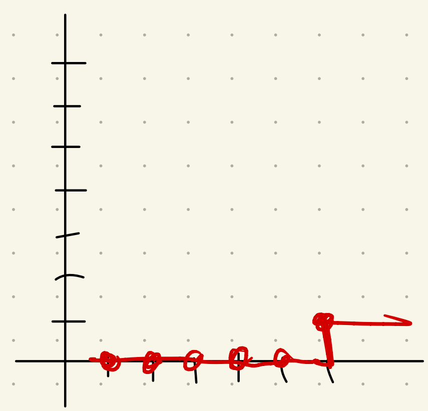
What are β_0 & β_1 curves here?

β_0:

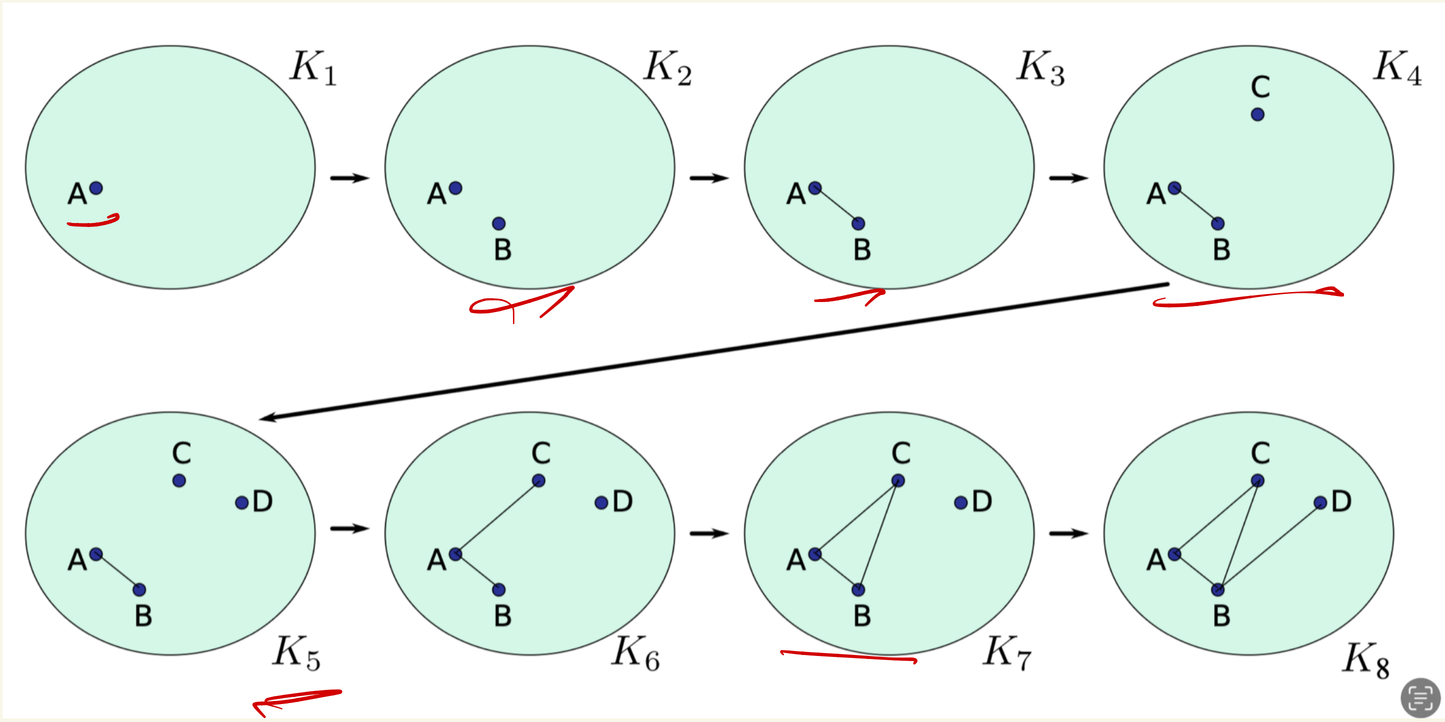


rank homology of K_i

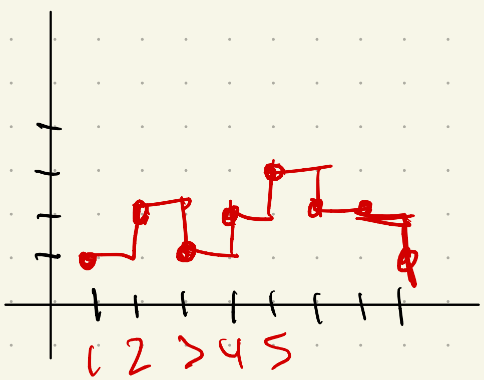
β_1:



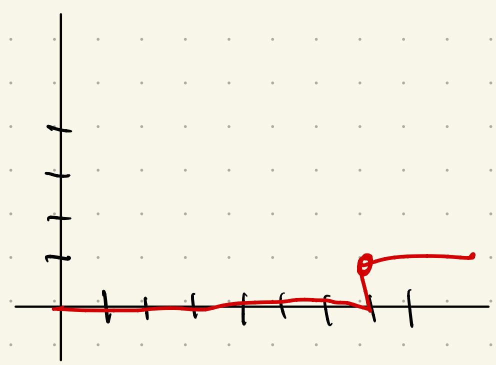
Another example



β_0



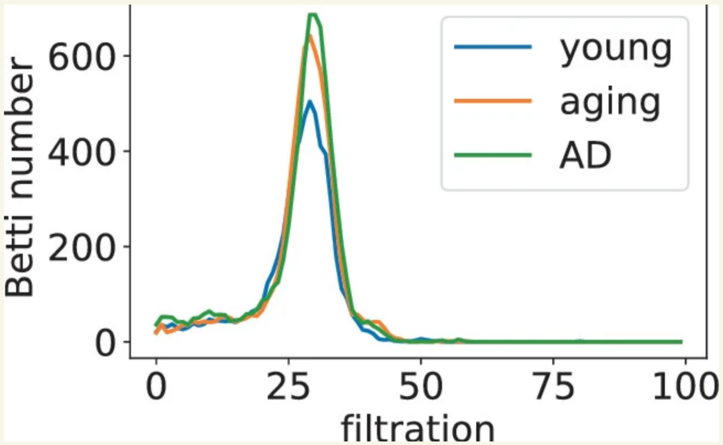
β_1



These are useful!

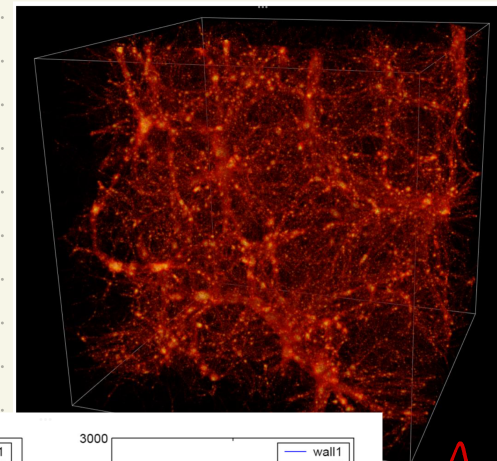
Detecting Alzheimer's

Saadat-Yazdi 2021



Cosmic Web

Weygaert et al 2011
Tymchyshyn et al 2023



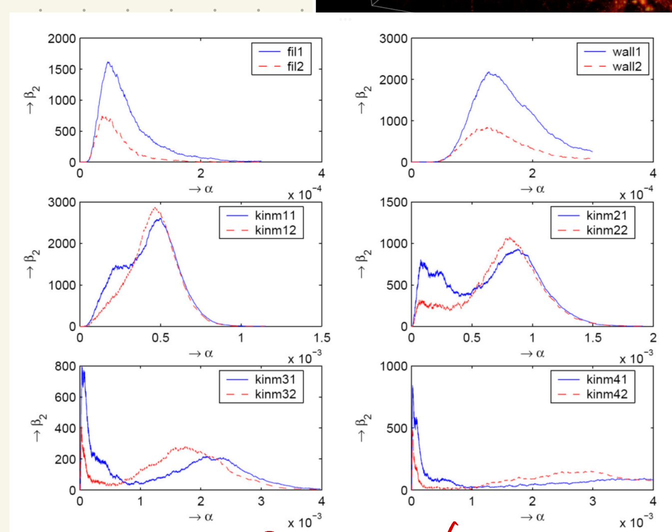
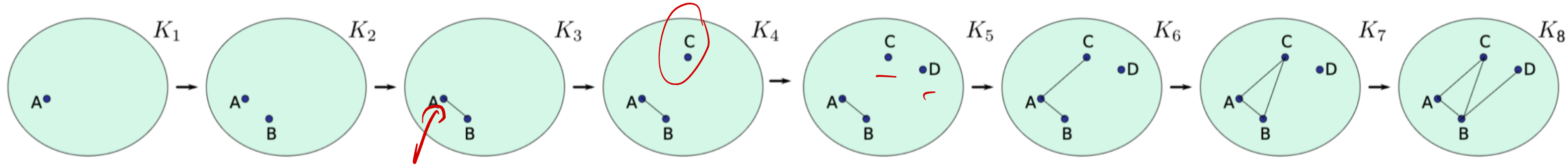


Figure 8. The dependence of the second Betti number, β_2, on α, for six different Voronoi clustering models. Top left: Voronoi filament model. Top right: Voronoi wall model. Centre left to bottom right: four stages of the Voronoi kinematic model, going from a moderately evolved model dominated by walls (center left) to a highly evolved model dominated by filaments and clusters (bottom right). Blue lines: realizations with 8 nuclei or cells. Red lines: realizations with 64 nuclei or cells.

(Many More...)

Now let's try tracking generators of homology!

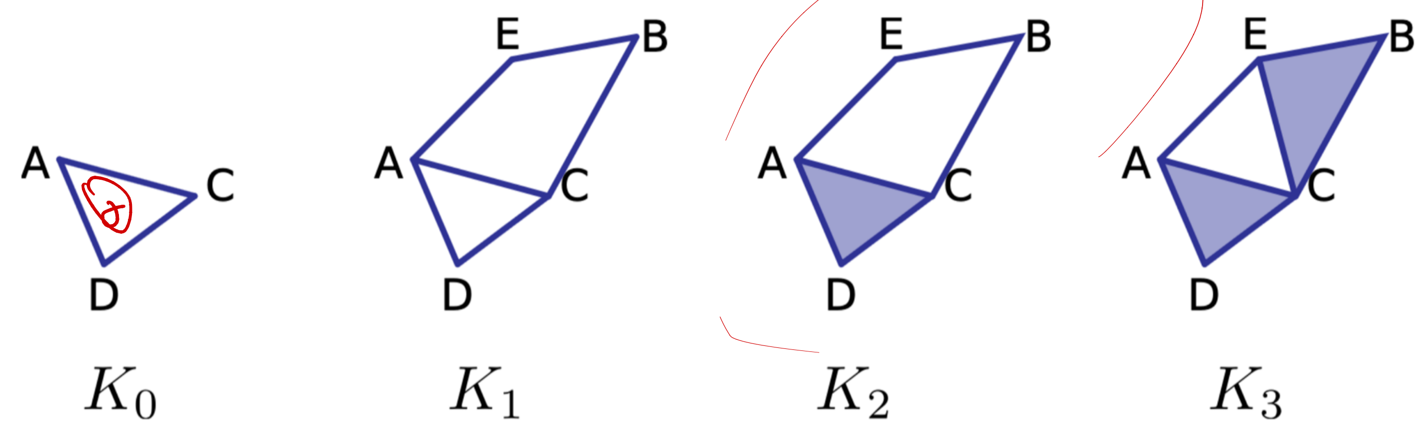


$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$$\begin{tikzcd}
{[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \\
& {[B]} \arrow[ur] & & {[C]} \arrow[r] & {[C]} \arrow[ur] & {[D]} \arrow[r] & {[D]} \arrow[ur] & \\
& & & & {[D]} \arrow[ur] & & &
\end{tikzcd}$$

$$\begin{array}{cc} & {[A]} \\ \begin{matrix} {[A]} \\ {[B]} \end{matrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array}$$

Another:



$$H_1(K_0) \xrightarrow{f_*} H_1(K_1) \xrightarrow{g_*} H_1(K_2) \xrightarrow{h_*} H_1(K_2)$$

$$\begin{tikzcd}
0 & 0 & 0 & 0 \\
{[ACD]} \arrow[r] & {[ACD]} \arrow[ur, bend left] & {[ABCE]} \arrow[r] & {[ACE]} \\
 & {[ABCE]} \arrow[ur, bend left] & &
\end{tikzcd}$$

The p^{th}-persistent homology groups are the images induced by inclusion:

$$H_p^{i,j} = \operatorname{Im}\left(H_p(K_i) \longrightarrow H_p(K_j)\right)$$

$K_i \subseteq K_j \quad i \leq j$

The p^{th}-persistent Betti numbers are

$$\beta_p^{i,j} = \operatorname{rank}\left(H_p^{i,j}\right)$$

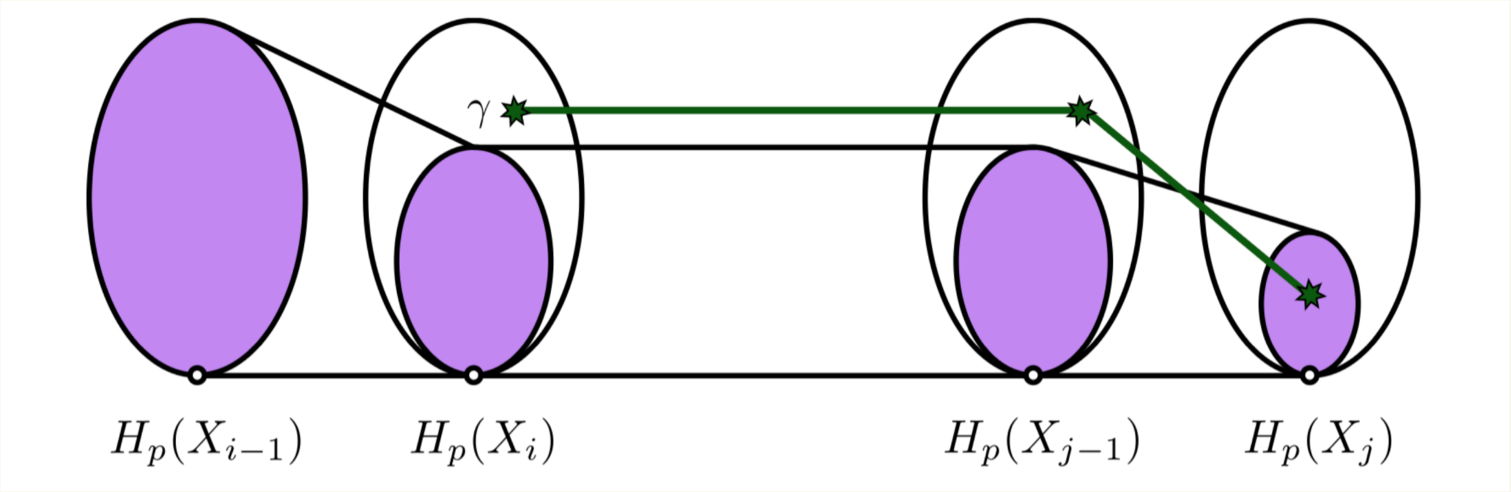
for a persistence module

$$H_p(K_0) \longrightarrow H_p(K_1) \longrightarrow \cdots H_p(K_i) \longrightarrow \cdots H_p(K_j) \longrightarrow \cdots H_p(K_n)$$

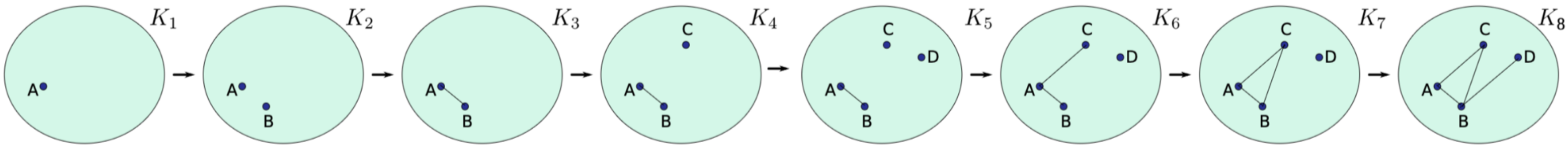
Birth & death

We say a homology class $\gamma \in H_p(K_i)$ is **born** at K_i if it is not in $H_p^{i-1,i}$, & γ dies entering K_j if it merges with an older class, ie if $f_p^{i,j-1}(\gamma) \notin H_p^{i-1,j-1}$ but $f_p^{i,j}(\gamma) \in H_p^{i-1,j}$.

Warning: not book's version!



Revisiting: when are births & deaths?



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$$\begin{tikzcd}
{[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \arrow[r] & {[A]} \\
 & {[B]} \arrow[ur] & & {[C]} \arrow[r] & {[C]} \arrow[ur] & {[D]} \arrow[r] & {[D]} \arrow[ur] & \\
 & & & & {[D]} \arrow[ur] & & &
\end{tikzcd}$$

b is born

b dies

c is born

d is born

c dies

d dies

Next time:

Barcodes + persistence diagrams.