TDA - fall 2025

Discrete
Morse Theory

Recall: Morse theory

First few weeks of class, we discussed (continuous) Morse theory,

Let's quickly recall...

[Begin old slides]

Smooth

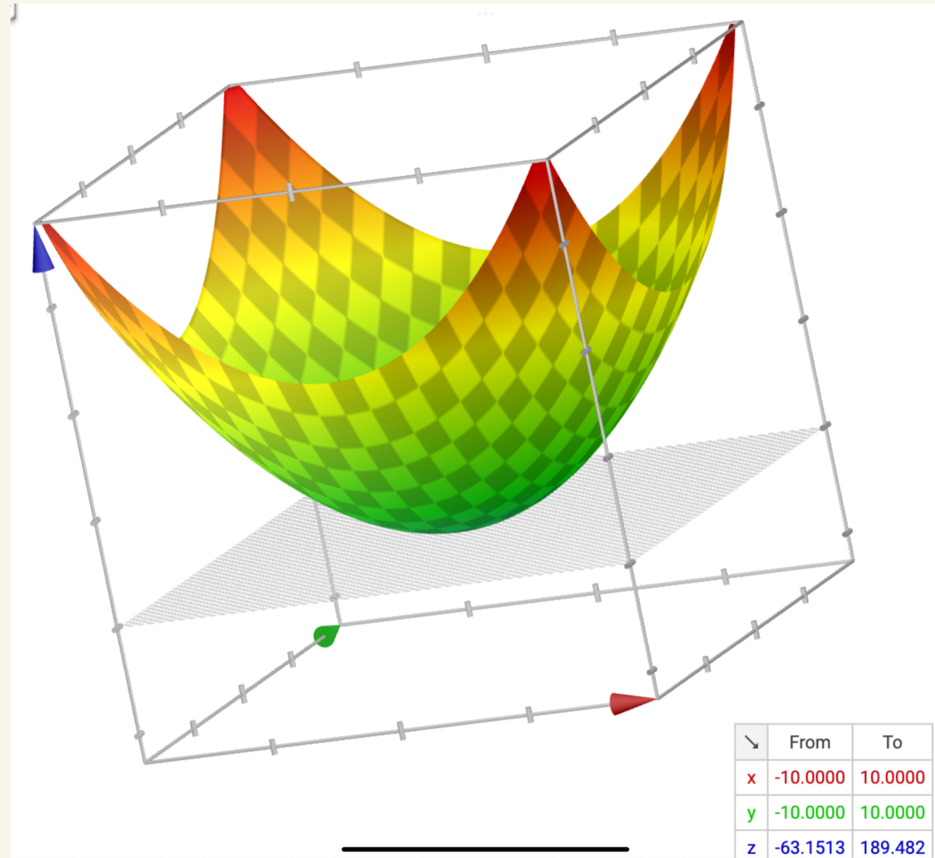
Topological manifolds are spaces
But usually, consider an embedding
into Euclidean space $\Rightarrow$ geometry.

Given a smooth function $f : \mathbb{R}^d \to \mathbb{R}$,
the gradient vector field $\nabla f : \mathbb{R}^d \to \mathbb{R}^d$
at a point x is:

$$\nabla f = \left[\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_d}(x) \right]$$

Ex: $f: \mathbb{R}^2 \to \mathbb{R}$

$$f((x_1, x_2)) = x_1^2 + x_2^2$$



$$\nabla f = \left[\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f\right]$$

$$= [2x_1, 2x_2]$$

Then $\nabla f((0,0)) = [0,0]$

$\nabla f((1,0)) = [2,1]$

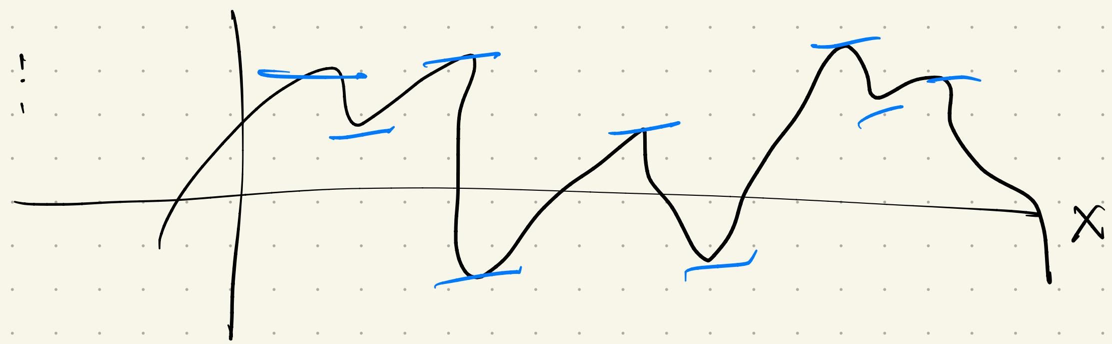
Critical point

Any $p \in \mathbb{R}^d$ where $\nabla f(p) = \vec{0}$

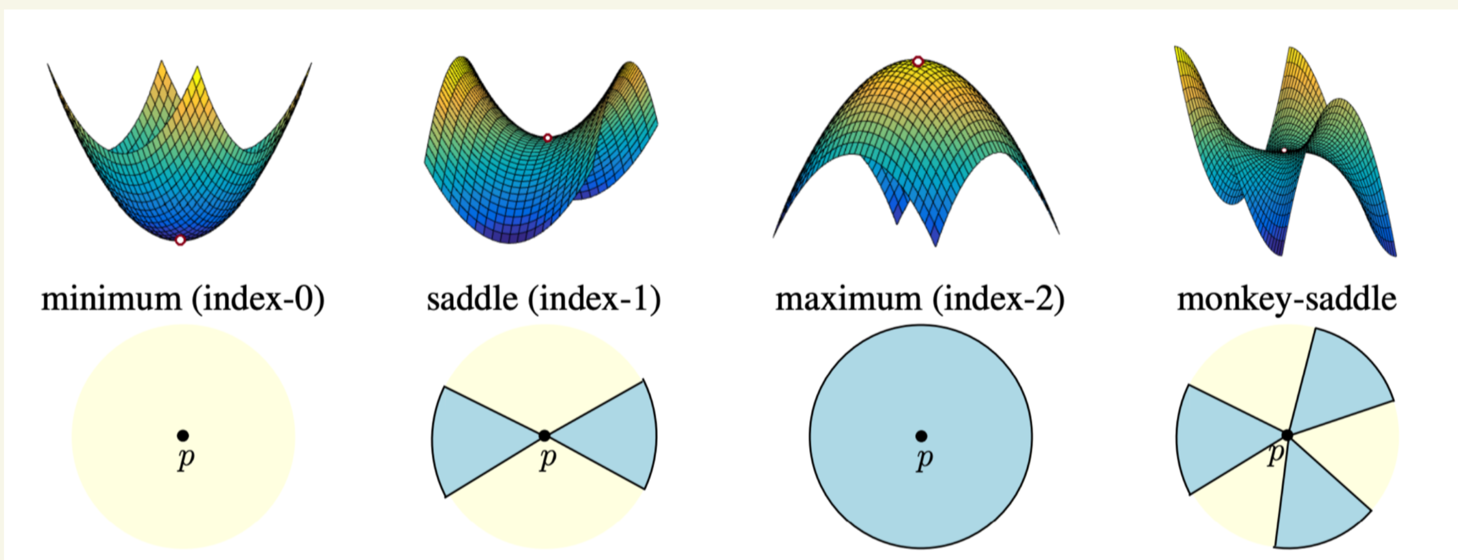
(Otherwise we say p is regular)

On 1 manifolds:

$$\frac{\partial f}{\partial x} \cdot x = 0$$



On 2 manifolds.



Extending to manifolds:

Given $\Phi : U \to W$, $U \subseteq \mathbb{R}^k$ & $W \subseteq \mathbb{R}^d$ open sets, where

$$\Phi(x) = (\phi_1(x), \ldots, \phi_d(x))$$

The Jacobian of Φ is a $d \times k$ matrix of partial derivatives:

$$\begin{bmatrix} \frac{\partial \phi_1(x)}{\partial x_1} & \cdots & \frac{\partial \phi_1(x)}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_d(x)}{\partial x_1} & \cdots & \frac{\partial \phi_d(x)}{\partial x_k} \end{bmatrix}$$

Types of critical points

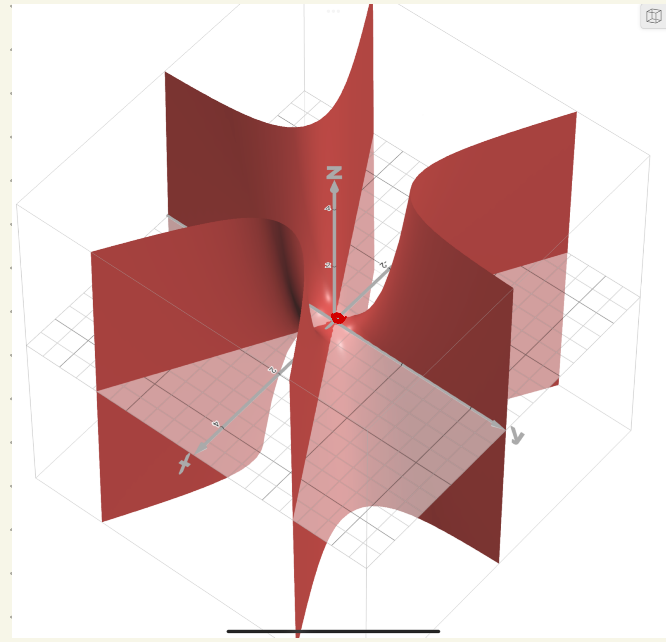
For a smooth m-manifold, the Hessian matrix of $f: M \to \mathbb{R}$ is the matrix of 2nd order partial derivatives:

$$Hessian(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(x) & \frac{\partial^2 f}{\partial x_m \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{bmatrix}$$

A critical point is non-degenerate if Hessian is nonsingular ($\det \neq 0$); otherwise degenerate.

An example: $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x_1, x_2) = x_1^3 - 3x_1x_2^2$$



$$\nabla f = \left[3x_1^2 - 3x_2^2,\ -6x_1x_2\right] \overset{(0,0)}{\Longrightarrow} [0, 0]$$

Is it degenerate?

Hessian:

$$\begin{pmatrix} \frac{\partial}{\partial x_1 \partial x_1} & \frac{\partial}{\partial x_1 \partial x_2} \\ \frac{\partial}{\partial x_2 \partial x_1} & \frac{\partial}{\partial x_2 \partial x_2} \end{pmatrix} = \begin{bmatrix} 6x_1 & -6x_2 \\ -6x_2 & -6x_1 \end{bmatrix}$$

So at $(0,0)$, det $= 0$

degenerate & critical at $(0,0)$

Morse Lemma

Given a smooth function $f: M \to \mathbb{R}$ defined on a smooth manifold M, let p be a non-degenerate critical point of f. Then $\exists$ a local coordinate system in a neighborhood $U(p)$ s.t.

- p's coordinanate is $\vec{0}$
- locally, any x is in the form

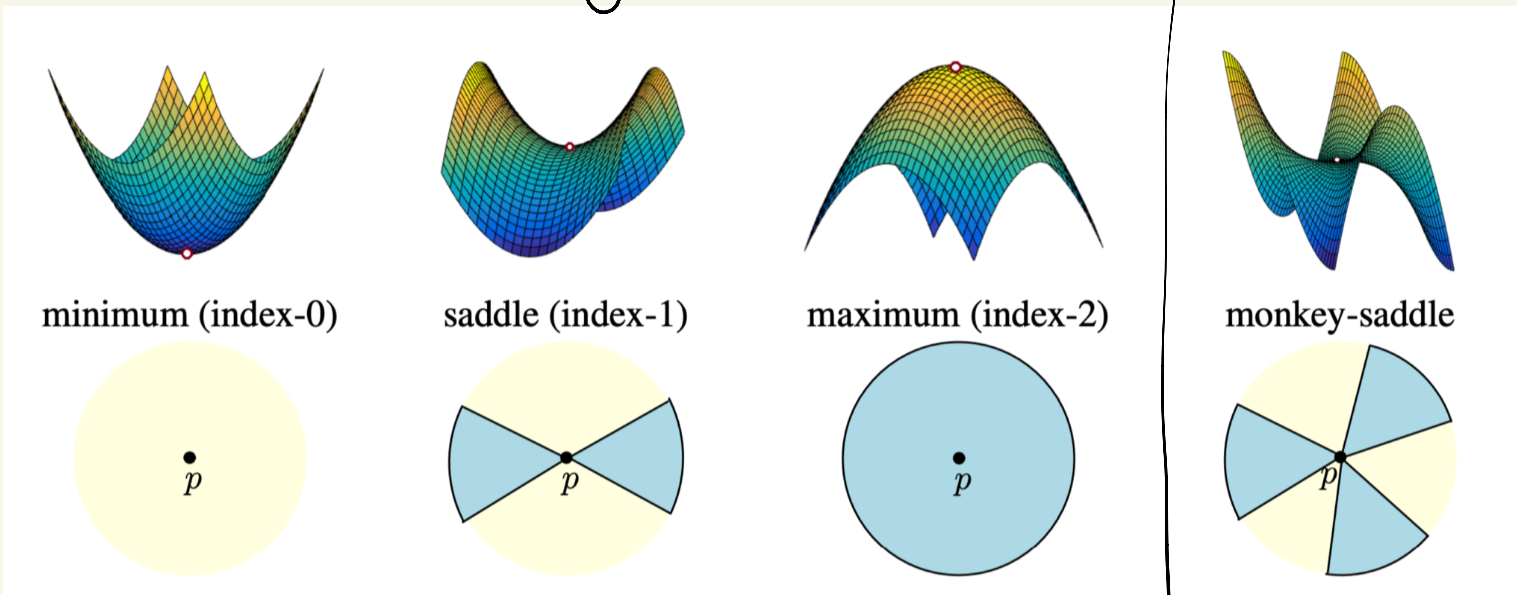
$$f(x) = f(p) - x_1^2 - \cdots - x_s^2 + x_{s+1}^2 + \cdots + x_m^2$$

for some $s \in [0, m]$

s is called the index of p.

Back to that picture...

non-degenerate | degenerate



everything is bigger around p

one coordinate bigger, one smaller

everthing is smaller around p

Morse functions

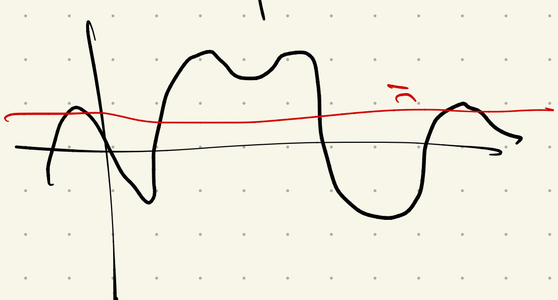
A smooth function $f: M \to \mathbb{R}$ (on a smooth manifold M) is a Morse function if

- none of f's critical points are degenerate
- the critical points have distinct function values

Some examples: Morse?

$f: \mathbb{R} \to \mathbb{R}$

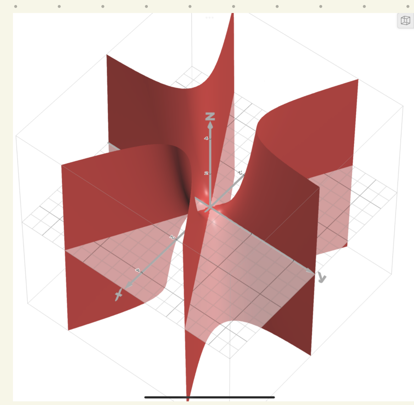
NO



$g: \mathbb{R}^2 \to \mathbb{R}$

$g((x_1, x_2)) = x_1^3 - 3x_1x_2^2$

NO



Begin new slides...

Discrete Morse theory

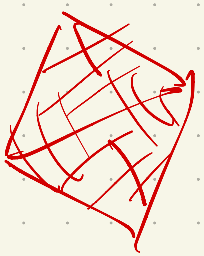
A few motivations:

- Attempt to simplify representation into a combinatorial format
  - ↳ Why? computation
    - ↳ smaller matrix
- Provide a tool to simplify simplical complexes
  - ↳ Why? Numeric calculations

A discrete Morse function f on a complex K is a function $f: K \to \mathbb{R}$ s.t. for every p-simplex $\sigma \in K$,

σ:

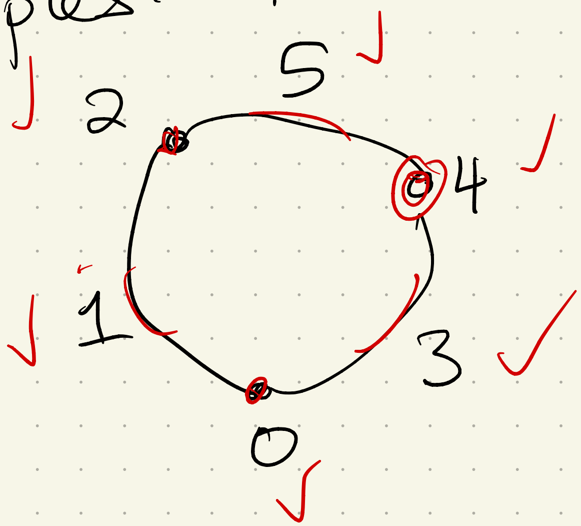
$$\left|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}\right| \leq 1$$

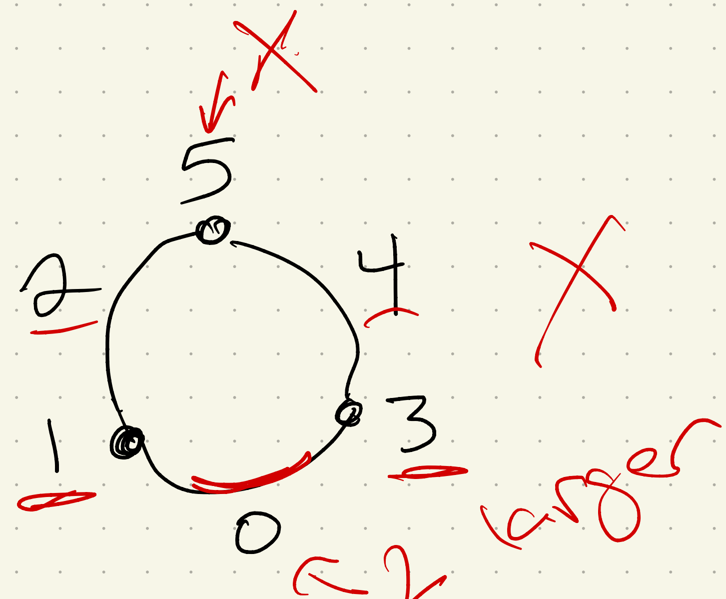


and

$$\left|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}\right| \leq 1$$

Examples: Yes/no?

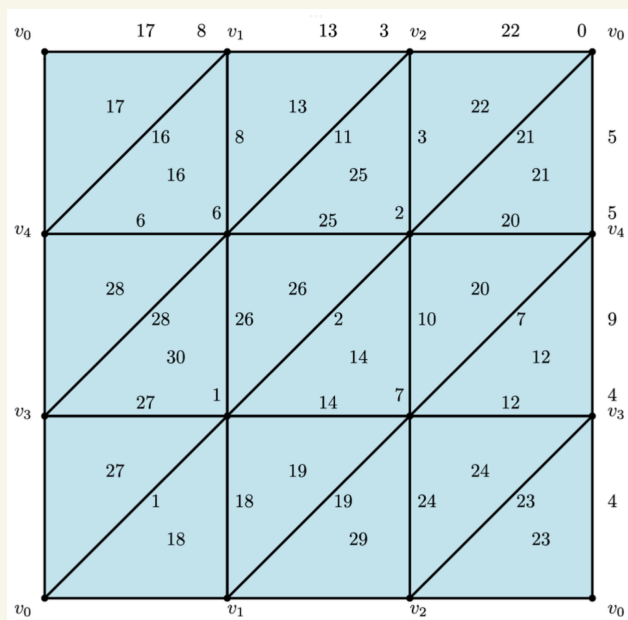




In other words,

- higher dimensional neighbors have higher values (with ≤ 1 exception)
- lower dimensional neighbors have lower values (with ≤ 1 exception)

Example:
9 vertex triangulation of the torus

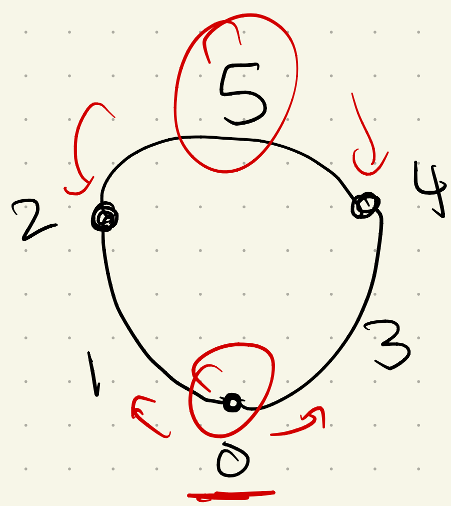


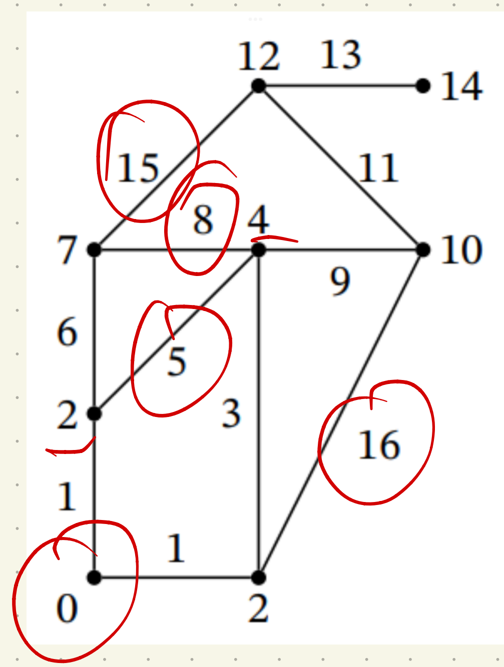
Critical Simplices

A p-simplex is critical with respect to f if $|\{\tau^{(p-1)} < \sigma : f(\tau) \geq f(\sigma)\}| = 0$

<u>and</u> $|\{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\}| = 0$

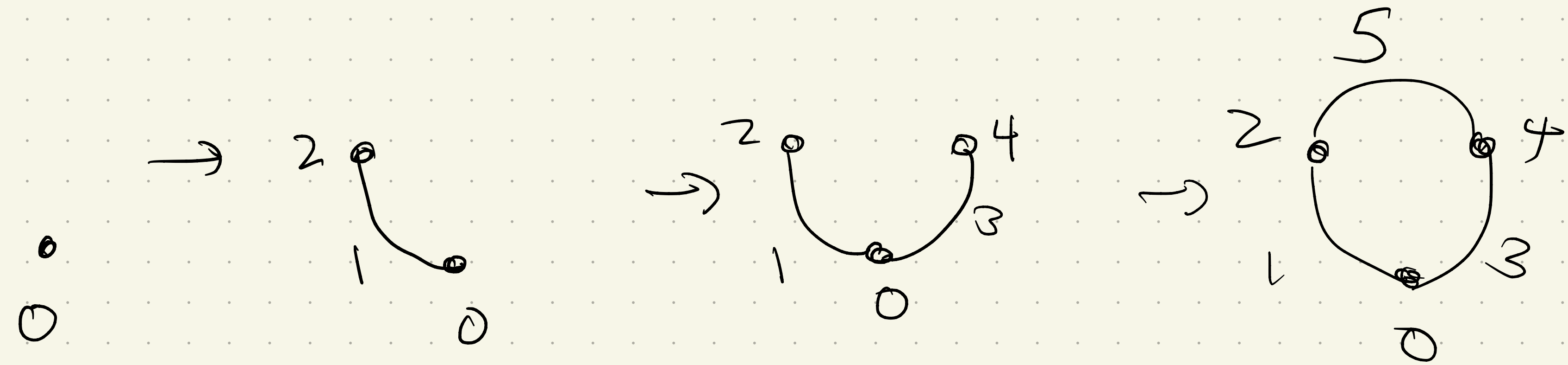
Example:





Why is this intuitive?

Think of levelsets & filtrations again:



- If $f(e) > f(v), f(w)$, with $e = vw$, then e connects two existing vertices
  ↳ changes homotopy type
- If $f(v) < f(e)$ for all incident edges e, then v is a new vertex not on any existing edge ⟶ changes homotopy type.

Regular points + discrete gradients

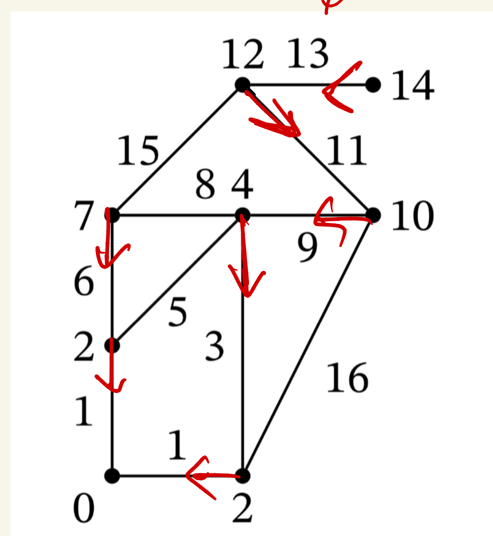
Any simplex that is not critical is regular, & will have one higher dim incident simplex with lower value or one lower dim simplex with higher.

pair ⇓

Discrete Gradient Vector field:

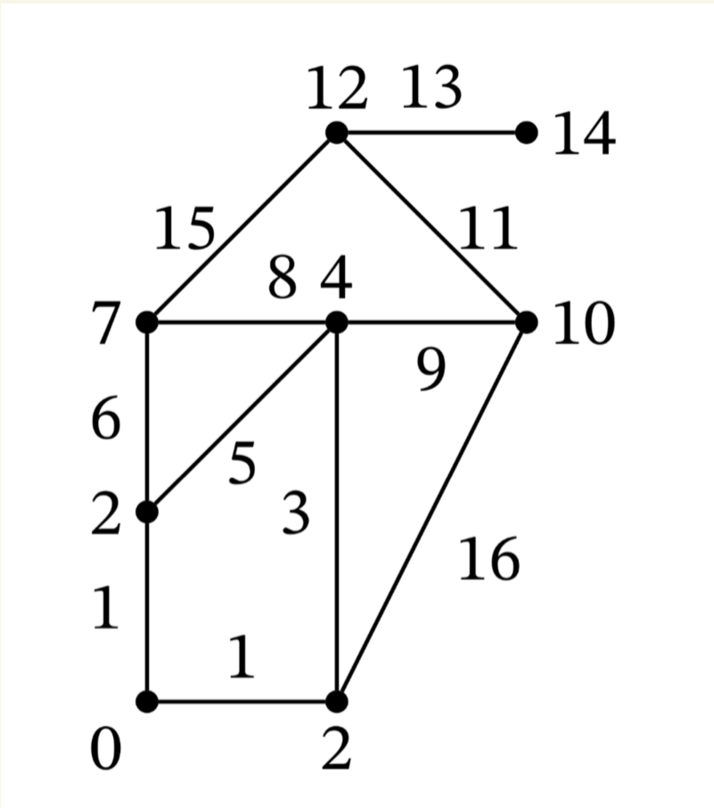
Draw arrow from σ to higher dim nbr with lower value

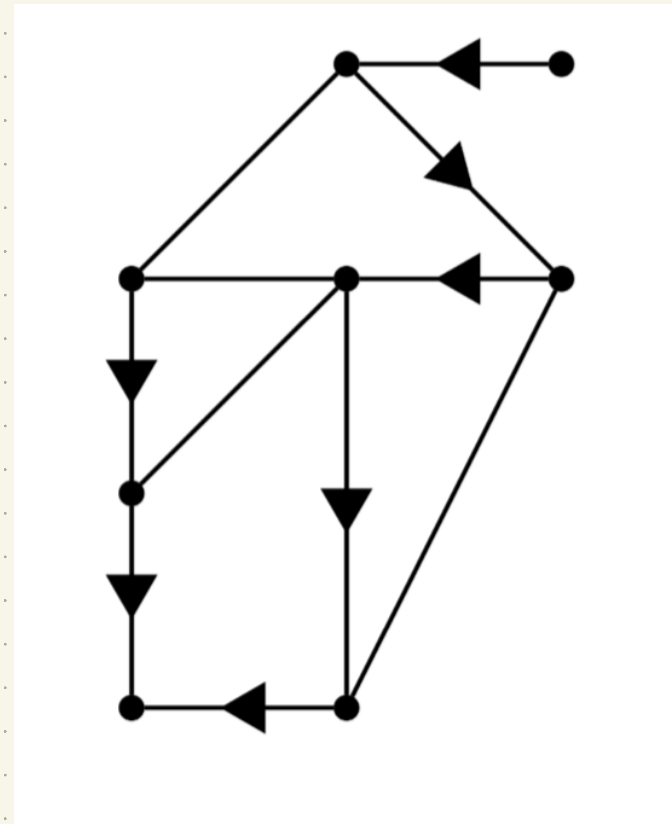
function f



V_i

Result: (check):

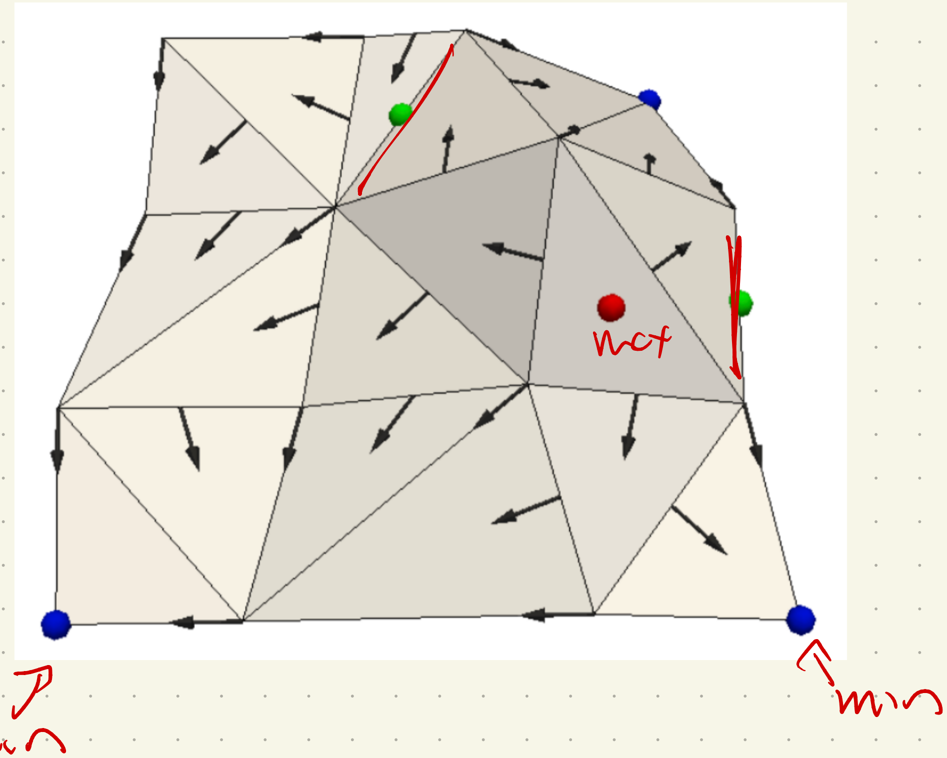




flow down wrt f

Result:

- Each simplex "flows" to at most one nbr
- Flow lines go down



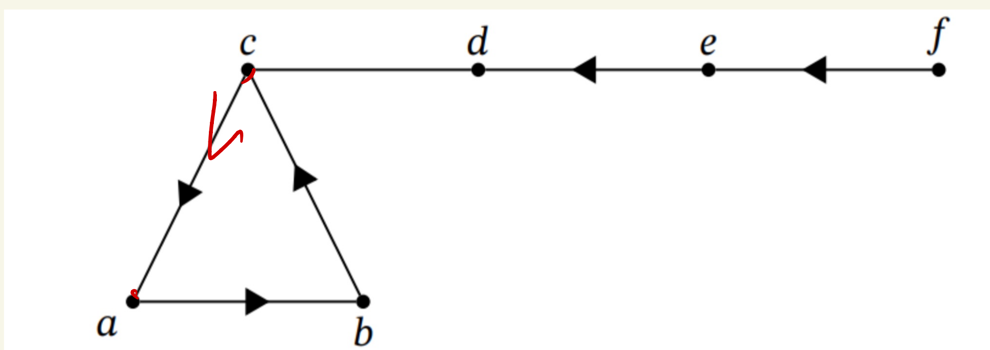
- Flow vanishes at critical simplices
  - ↳ Why? not paired

Definition (more general)

A **discrete vector field** ← not gradient on K is a collection of pairs $\{\alpha^{(p)} < \beta^{(p+1)}\}$ such that each simplex is in at most one pair.

Question: Are these the same as gradients?

No



need f:

$f(c) > f(ca)$

$f(a) > f(ab)$

$f(b) > f(bc)$

V-path:

sequence of simplices

$$[\tau_{-1}^{(p+1)}]\ \alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \ldots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}$$

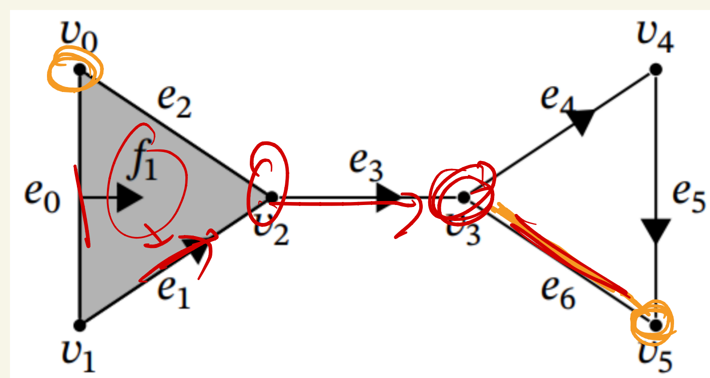
τ in field

such that $\alpha_i < \beta_i$ & $\beta_i > \alpha_{i+1} \neq \alpha_i$

& either τ_{-1} is critical or α_0 is regular

Ex:

$e_0 \to f_1 \to e_1 \to v_2 \to e_3 \to v_3$



maximal: go as far as possible

Theorem:

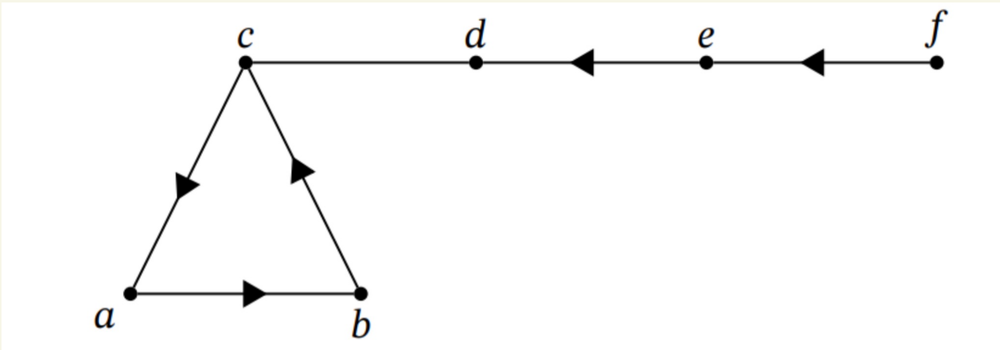
A discrete vector field V is the gradient of a discrete Morse function

$\iff$

no non-trivial closed V-paths.

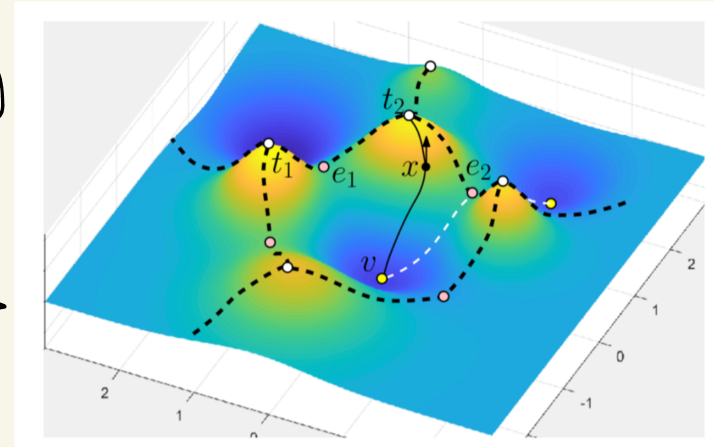
<u>Proof</u>

(by picture)



Continuous Morse functions

In continuous Morse theory, integral flow lines start & end at critical points



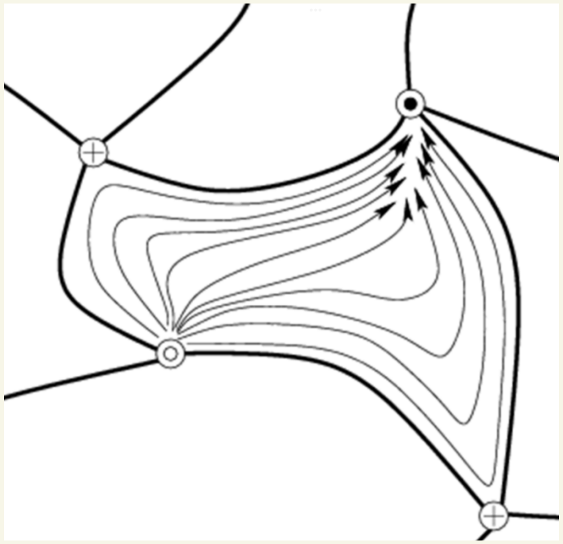
Why? derivative is 0

For a critical point p, p ∪ all integral lines into p = stable manifold

& all flow lines out of a critical point q is unstable manifold

[also called ascending & descending manifolds]

Morse-Smale Complex



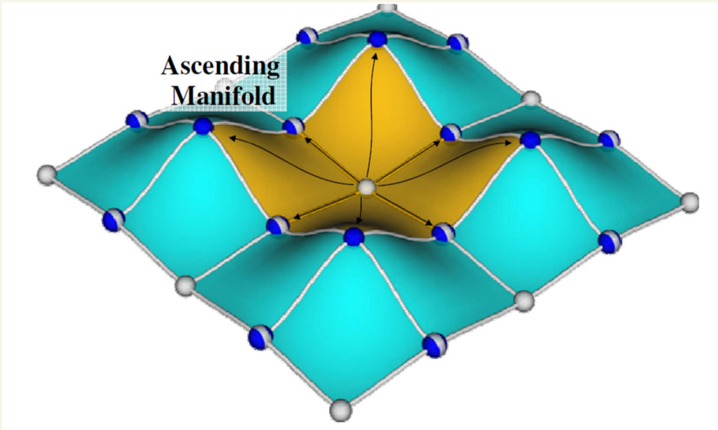
Partition manifold into cells, where each cell shares same source & destination points for integral lines

Vertices: critical points

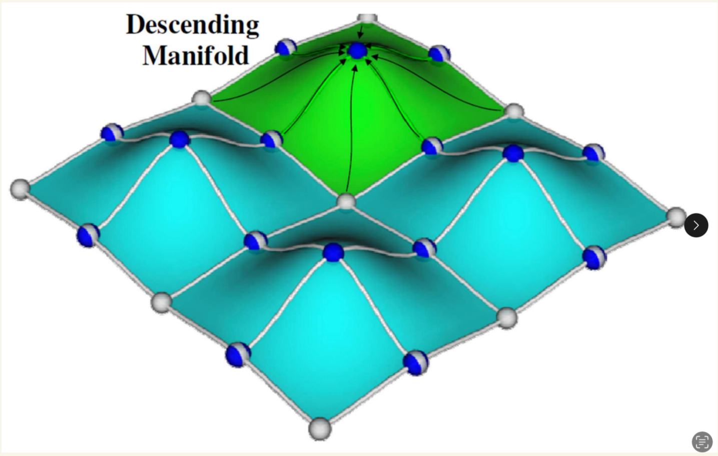
Arcs: integral lines connecting critical points

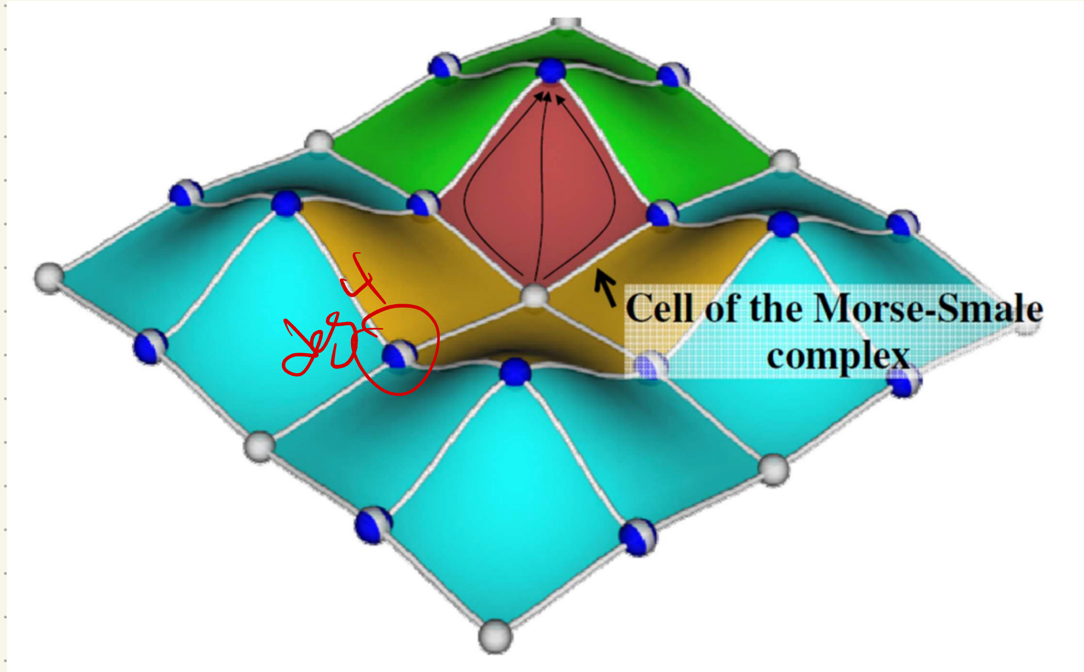
Faces: intersection of stable & unstable manifolds

For 2-manifolds: If generic, get:



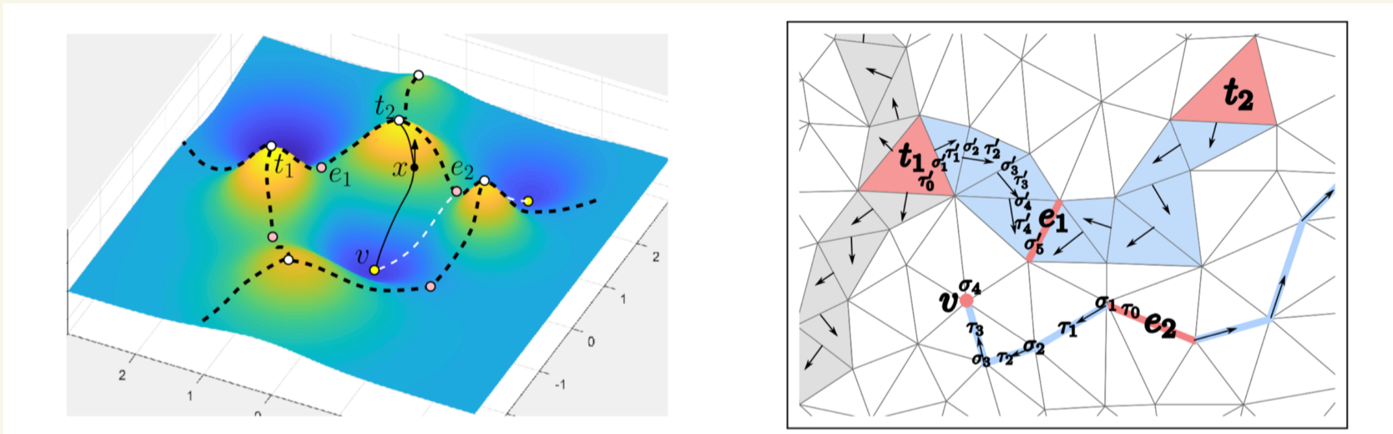
and





- Each saddle has 4 arcs (if generic)
- Each face: max-saddle-min-saddle
  "length" 4

Back to V-paths: either face-edge or edge-vertex

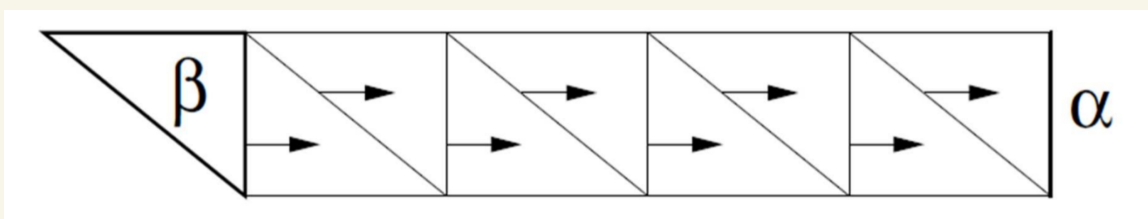


(a) (b)

continuous flow lines

discretized V-paths

ie:



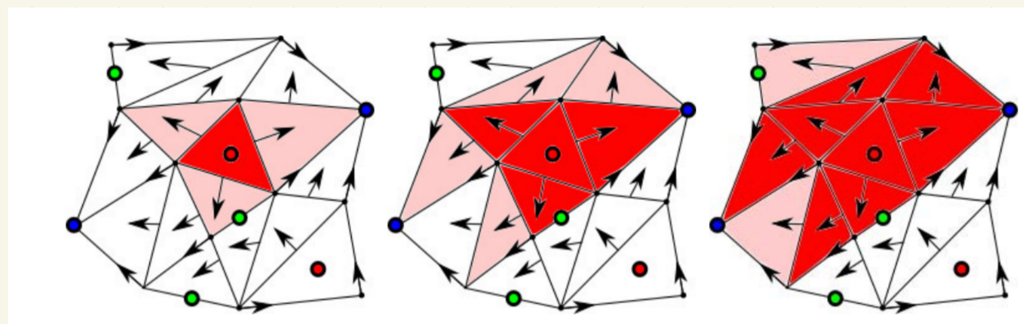
one difference: discrete flow goes down (not up like continuous)

So, in discrete setting: For critical edge e:

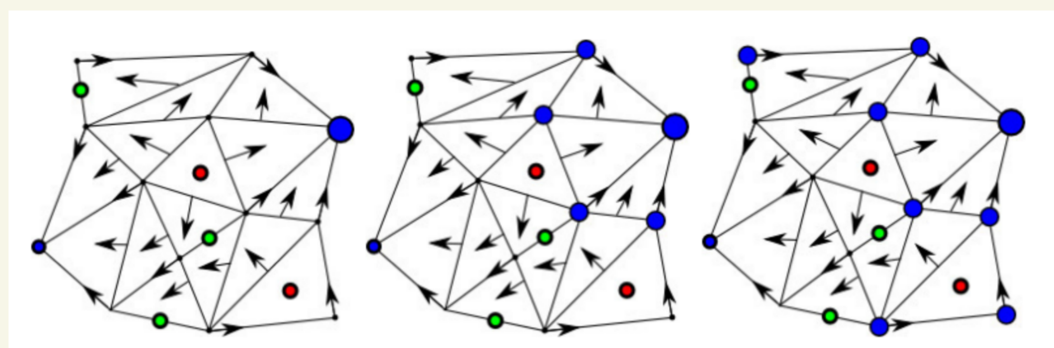
- stable manifold is union of edge-triangle gradient paths
- unstable manifold is union of vertex edge gradient paths

ie:

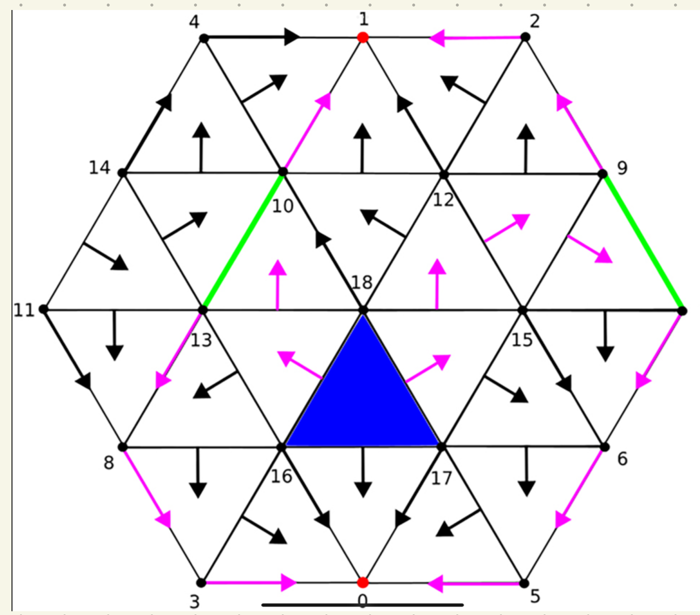
descending:



ascending:



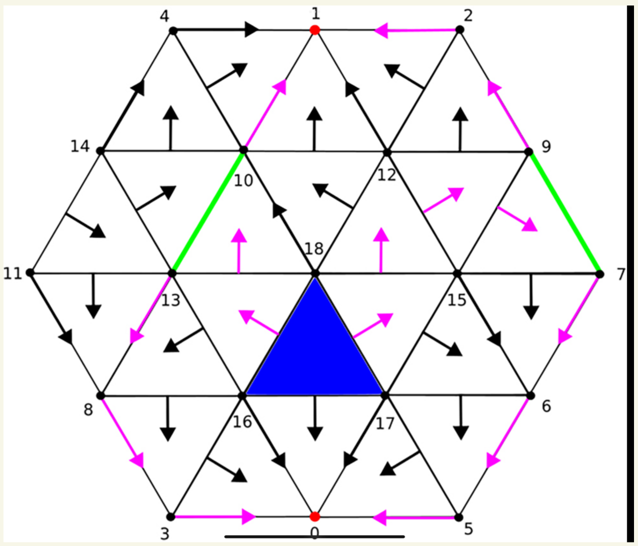
Separatrices:
V-paths between critical simplices (marked pink)



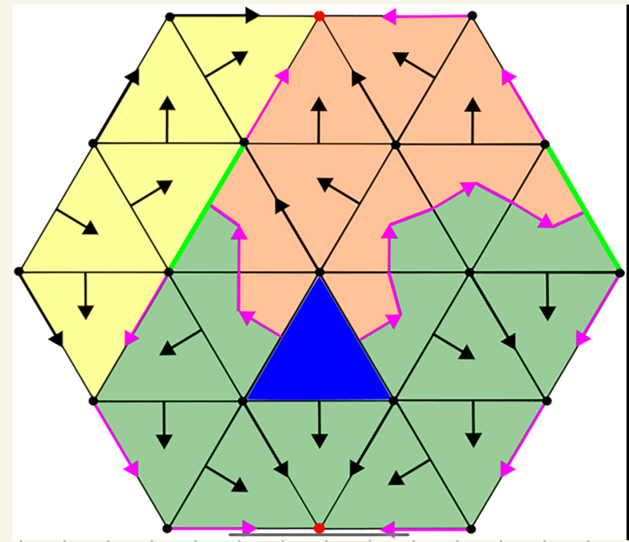
How to find?

- easy starting from critical edges
- From critical faces: try all options

Discrete Morse-Smale Complex



$\Rightarrow$



Consider Chain complex:

$$C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K)$$

Key: only critical simplices + sepratrices between them in ∂

... Why? $\rightarrow$

Back to motivation

In Foreman's original work, goal was to identify a simpler complex with same homology:

Let $M_p \subseteq C_p(K)$ be critical p-simplices

Then $\exists$ maps $\tilde{\partial}_p$ s.t. $\tilde{\partial}_{d-1} \circ \tilde{\partial}_d = 0$

with $M_d \xrightarrow{\tilde{\partial}_d} M_{d-1} \xrightarrow{\tilde{\partial}_{d-1}} \cdots \xrightarrow{\tilde{\partial}_1} M_0$

s.t. $H_d(M, \tilde{\partial}) \cong H_d(K)$

[next time → why, plus connection to persistence + applications]