TDA - fall 2025

Directional
Transforms

- Proposals → please email to me

- Next Wed: guest lecture

Recap

Last time, we saw time-varying functions which yielded a family of persistence diagrams.

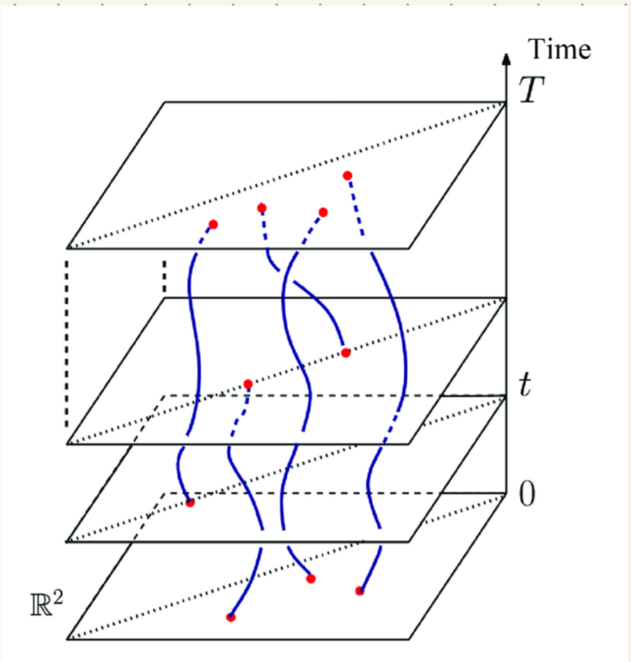
Stability $\Rightarrow$ can connect points between the diagrams

$$V(X) = \{ D(X(t)) \mid t \in [0,1] \}$$

$$[0,1] \to \mathbb{D} \leftarrow \text{space of pers. diagrams}$$

Dgm

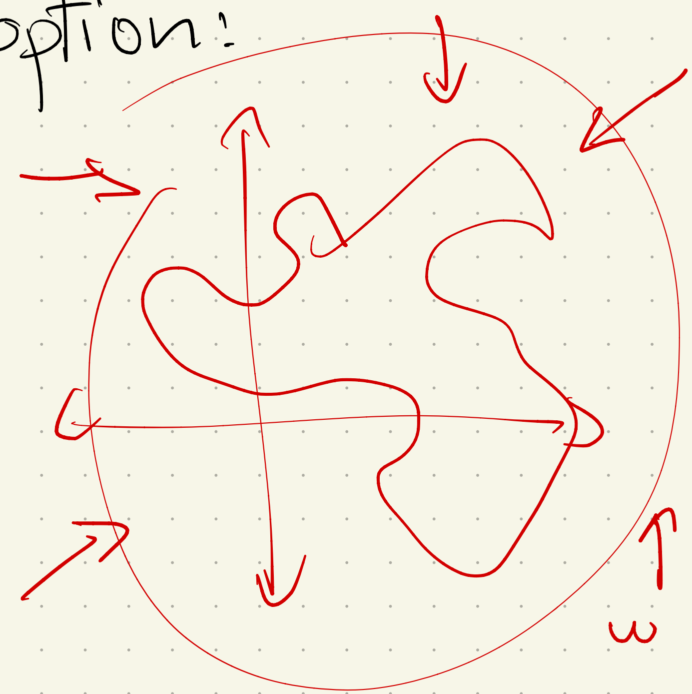
$$t \longmapsto D(X(t))$$



Natural question:

Are there good families of functions to study which capture features well?

Directional transforms are one increasingly popular option:



Shape $A \subseteq \mathbb{R}^d$

take $\omega \in S^{d-1}$

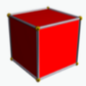
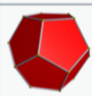
$\hookrightarrow f_\omega$ = height in ω's direction

$f_\omega(x) = \langle \omega, t \rangle$

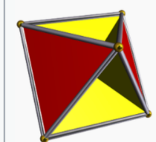
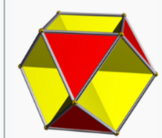
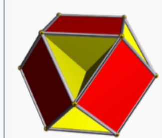
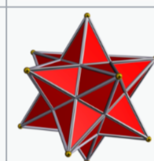
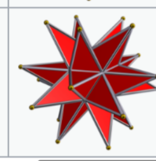
Recall: Euler characteristic

Vertices − Edges + Faces = χ

Convex:

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $\chi = V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

Nonconvex:

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $\chi = V - E + F$
Tetrahemihexahedron		6	12	7	**1**
Octahemioctahedron		12	24	12	**0**
Cubohemioctahedron		12	24	10	**−2**
Small stellated dodecahedron		12	30	12	**−6**
Great stellated dodecahedron		20	30	12	**2**

One of the earliest invariants.

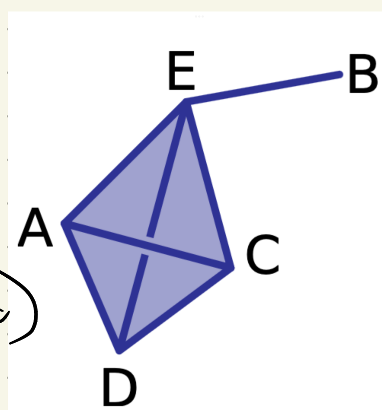
Generalized Euler Characteristic

For any simplicial complex K with n_p simplicies of dimension p, Euler characteristic is

$$\chi(K) = \sum_p (-1)^p n_p$$

Example

K:



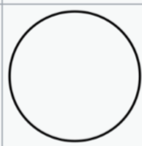
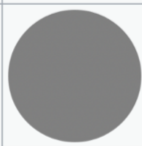
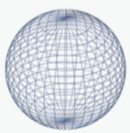
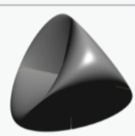
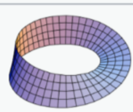
(filled in tetrahedra)

$$\chi(K) = -n_3 + n_2 - n_1 + n_0$$

$$= -1 + 4 - 7 + 5$$

$$= 1$$

Invariant on topological spaces

Name	Image	χ
Interval		1
Circle		0
Disk		1
Sphere		2
Torus *(Product of two circles)*		0
Double torus		−2
Triple torus		−4
Real projective plane		1
Möbius strip		0

<u>Theorem</u>:

For any simplicial complex K where n_p = # of p-dim simplices,

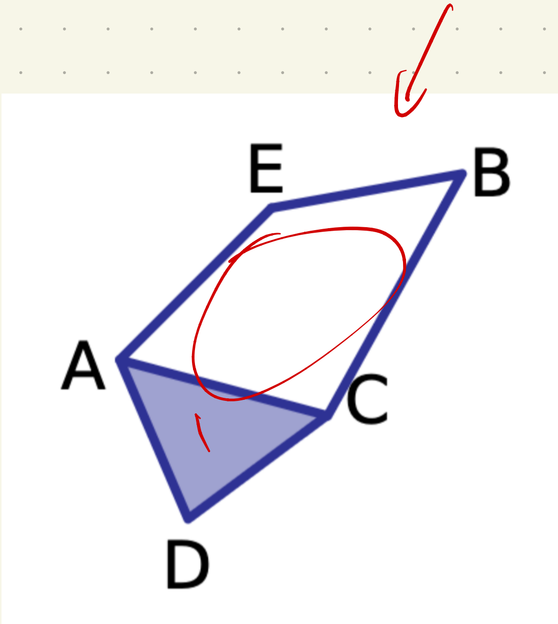
$$\chi(K) = \sum_p (-1) n_p$$

$$= \sum_p (-1)^p \beta_p$$

Betti #s

$\beta_p = \text{rank}(H_p)$

Check:



$$\sum_p (-1) \cdot n_p =$$

$$5-6+1=0$$

versus

$$\sum_p (-1) \cdot \beta_p = \beta_2 - \beta_1 + \beta_0$$

$$= 0 - 1 + 1 = 0$$

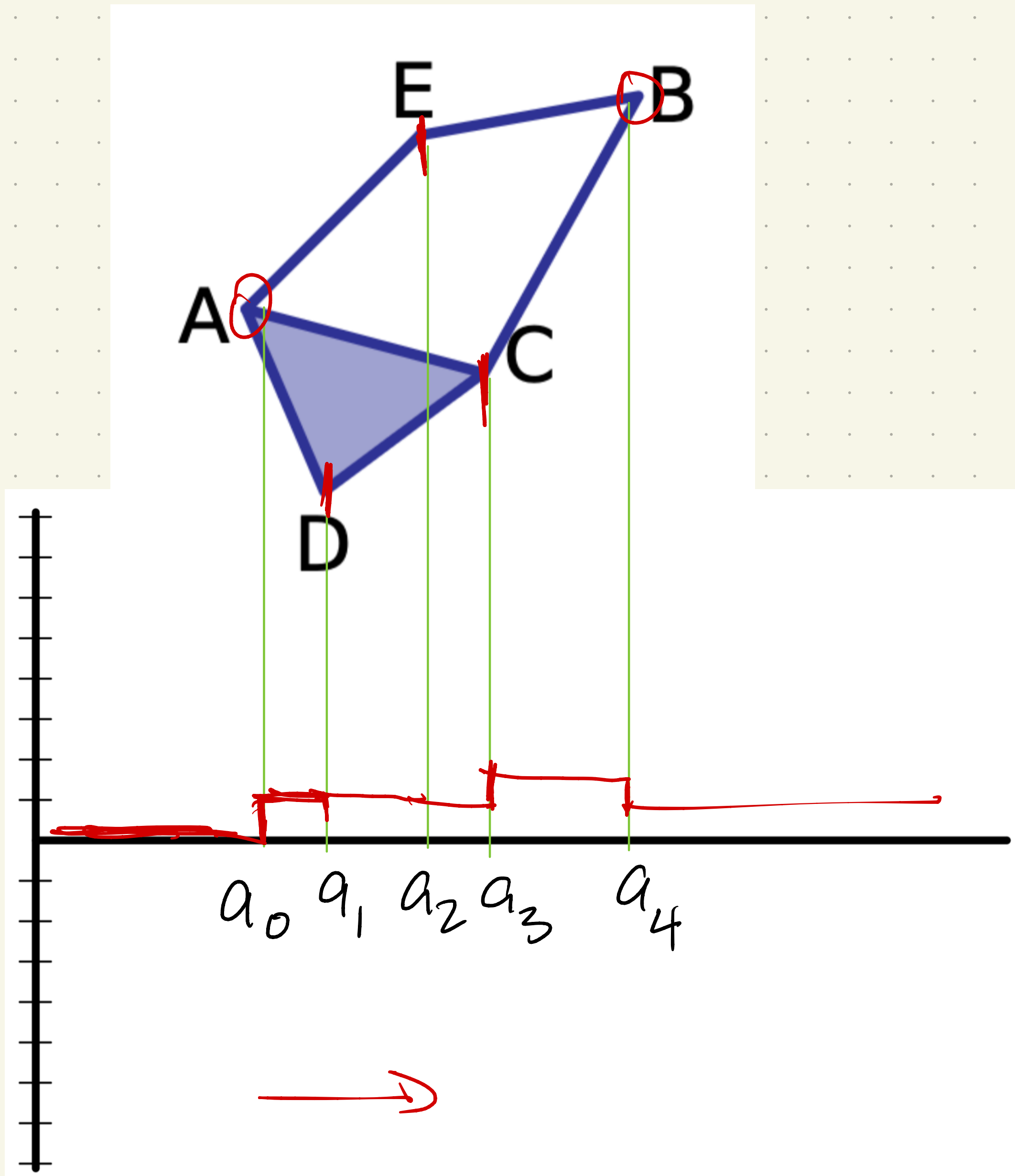
✓

Euler characteristic curve

Let $E(a) = \chi(f^{-1}(-\infty, a])$:

$\Downarrow$

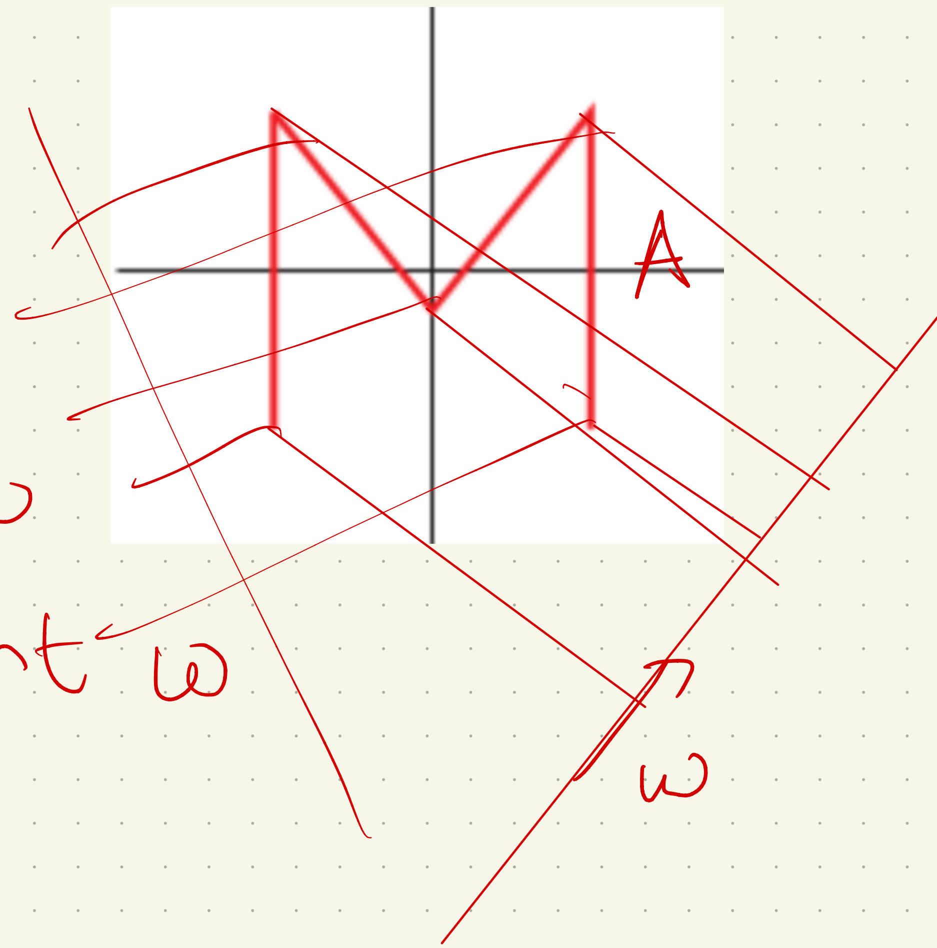
For each a, get a value, & then plot the curve



Euler transform

- Fix $A \subseteq \mathbb{R}^d$
- Fix direction $\omega \in S^{d-1}$
- Then $f_\omega : A \to \mathbb{R}$
  where $f_\omega(x) = \langle x, \omega \rangle$

in $\mathbb{R}^2$:



ie:

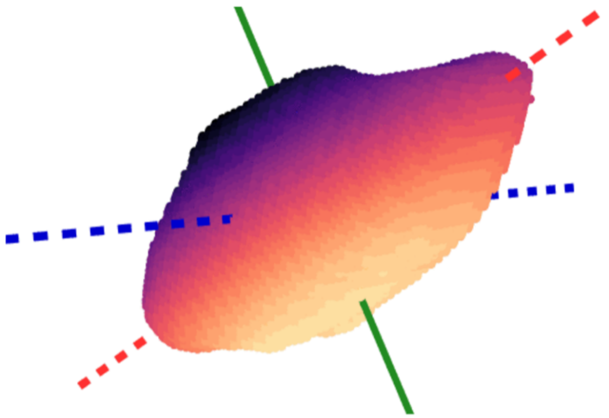
for any direction ω

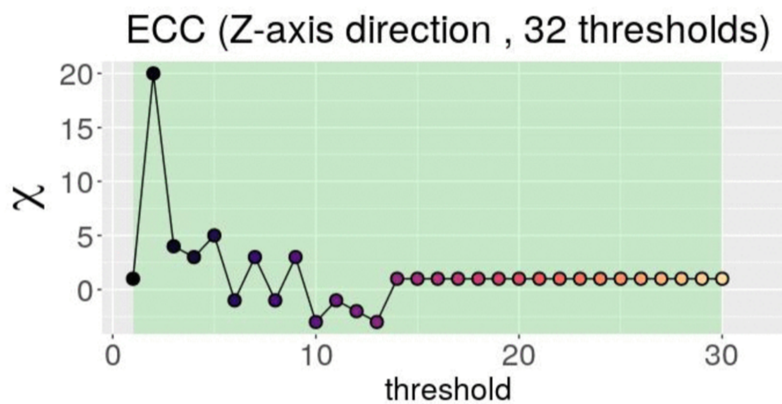
↳ Euler curve wrt ω

In 3d:

Euler Characteristic Curve

- Given embedded space $X \subseteq \mathbb{R}^d$
- Fix direction $\omega \in \mathbb{S}^{d-1}$
- Get function $f_\omega : X \to \mathbb{R}$, $x \mapsto \langle x, \omega \rangle$
- Get ECC $a \mapsto \chi f_\omega^{-1}(-\infty, a]$





(video from Amézquita et al 2021)

Transform Let M_d = space of all finite simplicial complexes in $\mathbb{R}^d$, and $A \in M_d$:

$$ECT(A): S^{d-1} \longrightarrow \text{fcns on } \mathbb{R}$$
$$\omega \longmapsto \chi_\omega(A)$$

Taking a step back, consider all such maps:

$$ECT: M^d \longrightarrow \text{Fcns from } S^{d-1} \text{ to Euler curve}$$

$$A \longmapsto (\omega \mapsto \chi_\omega(A))$$

In other words:

ECT takes a space, & gets all Euler curves for all possible directions.

Theorem: $\boxed{\text{Turner, Mukherjee, Boyer } \underline{2014}}$ ↳ $\boxed{\text{also } 2022}$

For compact, definable sets in $\mathbb{R}^d$, the map ECT is injective.

Translation: Given all directions, two shapes are identical if all Euler curves are identical.

Why is this surprising?

• Euler characteristic is lossy:

Recall

Name	Image	χ
Interval		1
Circle		0
Disk		1
Sphere		2
Torus *(Product of two circles)*		0
Double torus		−2
Triple torus		−4
Real projective plane		1
Möbius strip		0

all directions
↳ all sublevel sets

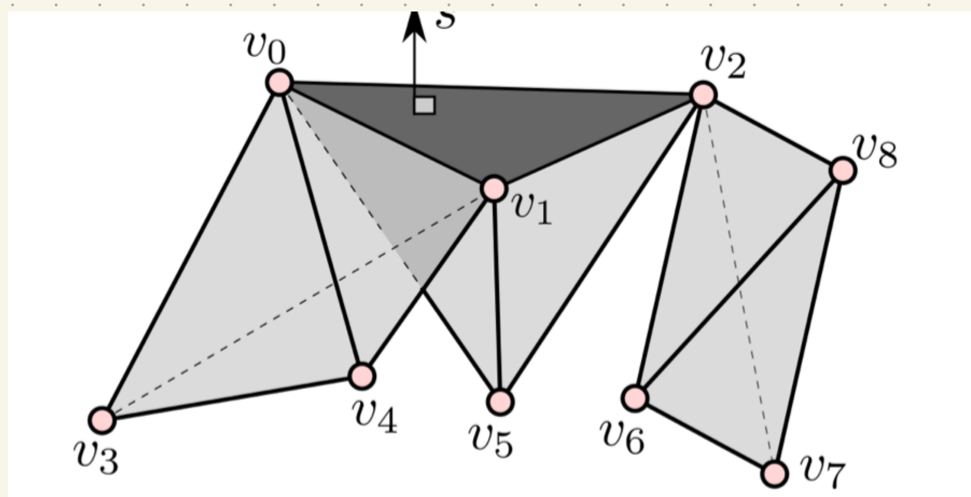
#

Proof idea (simplicial complex version)

For every simplex, need a direction where ECT "sees" the simplex:

not ∞

S^1



Fasy et al 2019

$\subset$ verbose PD

Note:

Original proof is not in simplicial setting.

How many directions?

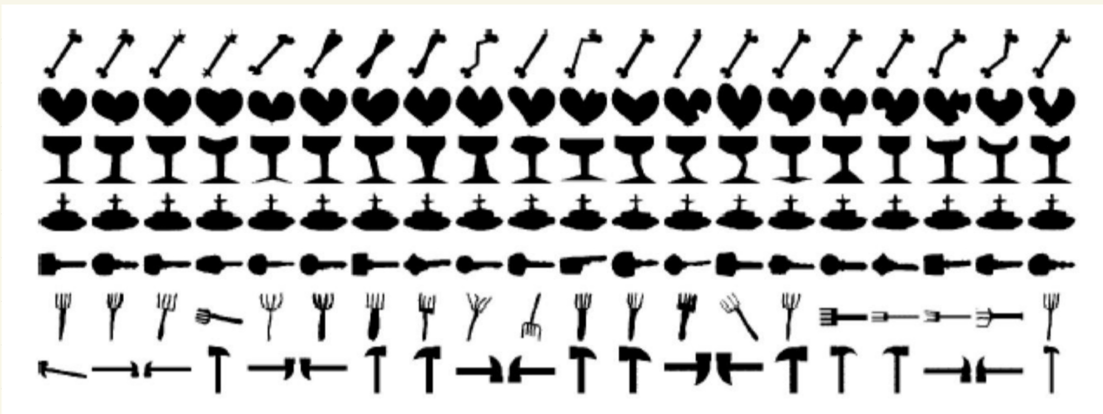
$\boxed{\text{Curry et al 2022}}$ give a bound based on the number of Euler critical values → exponential in dimension

↳ $\boxed{\text{Fasy et al 2019}}$ show they can use $O(n2^k + d)$ directions for n simplices embedded in $\mathbb{R}^d$ with K-dimensional complex.

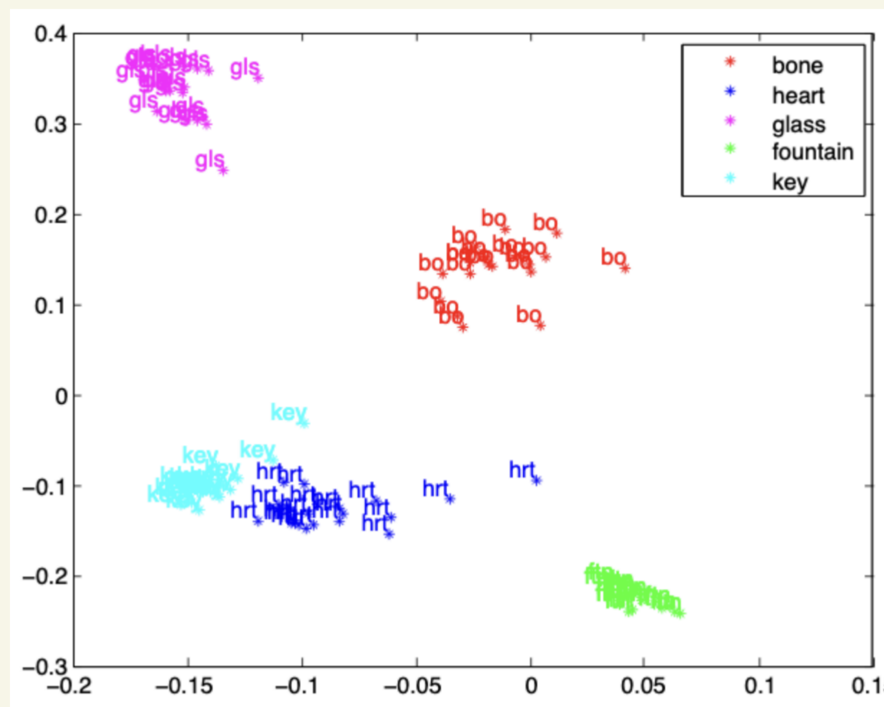
(technically using a slight variant called <u>verbose</u> ECT)

Initial proof of concept (2014 paper)

2 + 3d shapes → 64 directions



Works well!



In practice

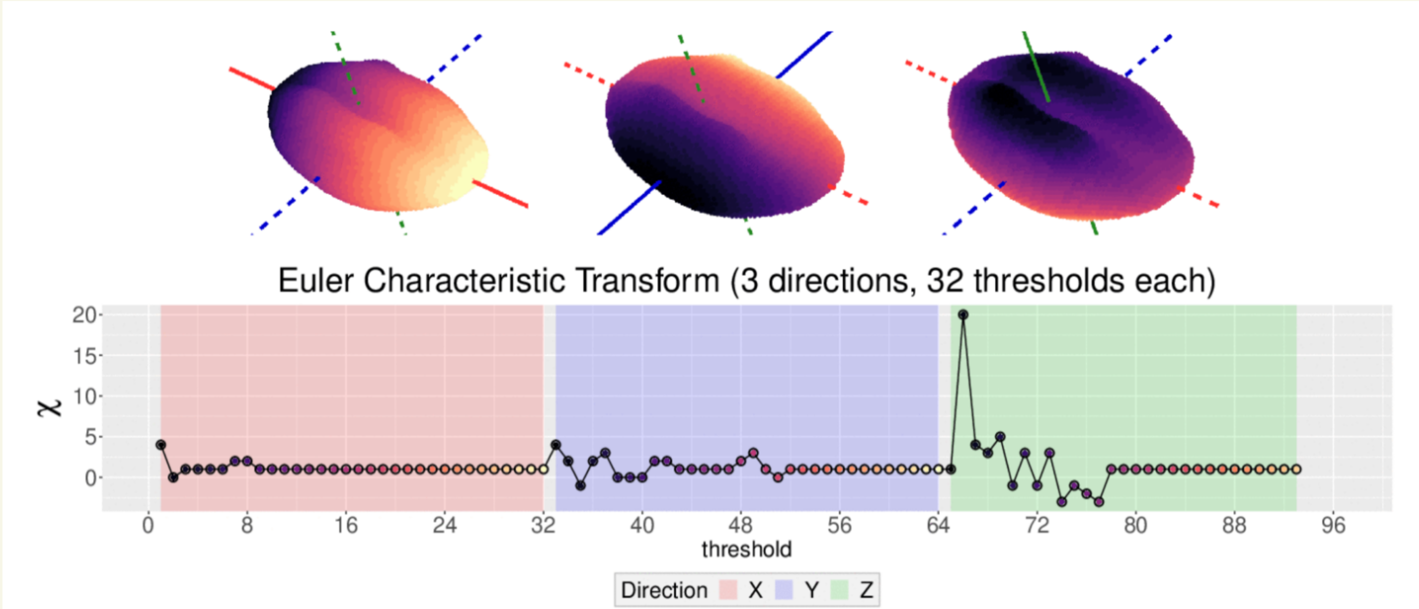
Many applications sample a small number of directions

↳ seems to work well!

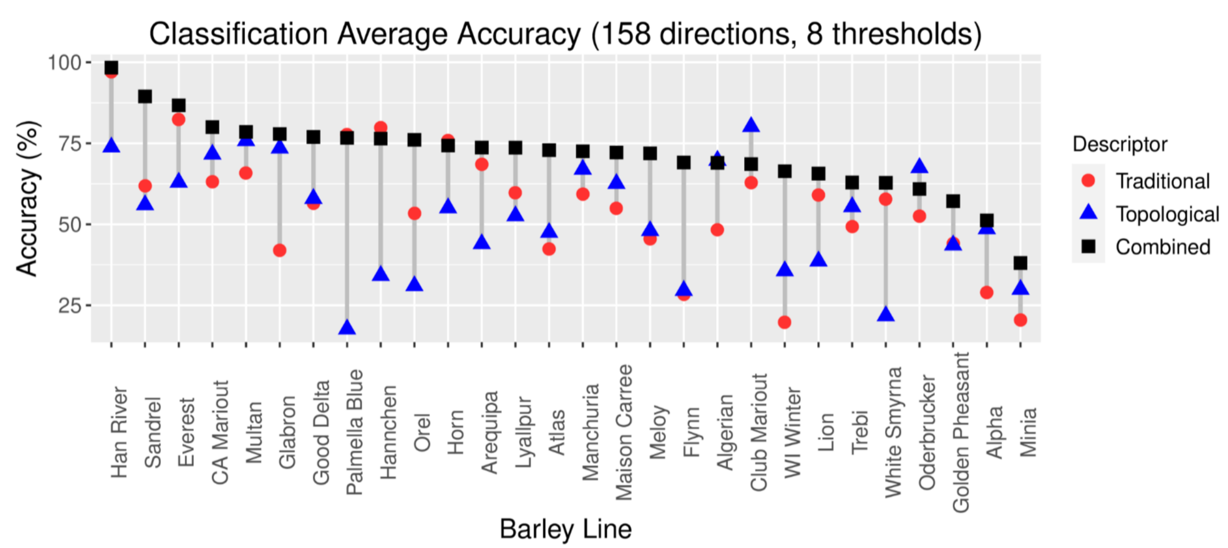
Example: Barley seed demo again (used 3 directions on relatively well-aligned data)

Compared to (& combined with) more traditional classifications

Results



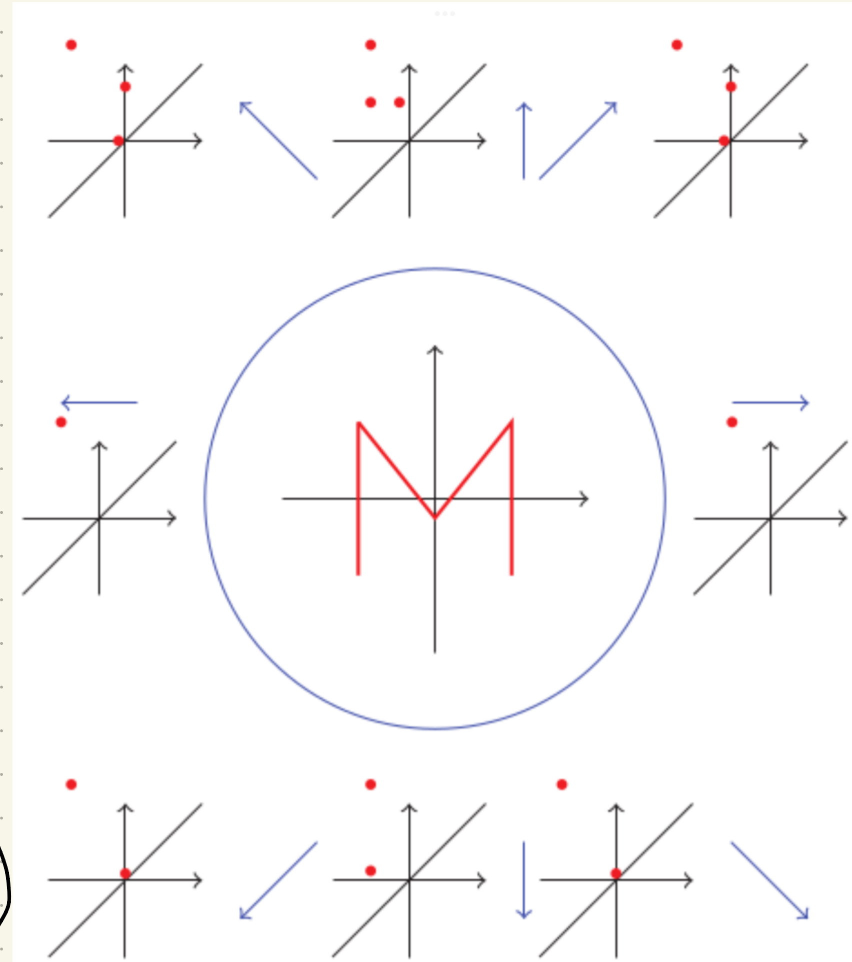
Shape descriptors	No. of descriptors	Precision	Recall	F1
Traditional	11	0.57 ± 0.058	0.56 ± 0.019	0.55 ± 0.019
Topological	12	0.51 ± 0.063	0.51 ± 0.020	0.50 ± 0.020
Combined	23	0.72 ± 0.055	0.71 ± 0.018	0.71 ± 0.018



Persistent Homology Transform

Same idea, but diagrams

$$PHT(A): S^{d-1} \to Dgm$$
$$\omega \mapsto PD_{f_\omega}(A)$$



and

$$PHT: M_d \to \left\{ \begin{array}{c} \text{fcns from} \\ S^{d-1} \text{ to } Dgm \end{array} \right\}$$

$$A \longmapsto \{\omega \mapsto PD_{f_\omega}(A)\}$$

Theorem

Curry et al 2022, Ghrist etall 2018

also 2014

The PHT is injective
(on "nice enough" spaces)

So: if Persistence diagrams from all directions are the same
$\Rightarrow$ shapes are the same.

[Again, quite surprising!]

(no lower bnd)

In practice, still need exponential number of directions.

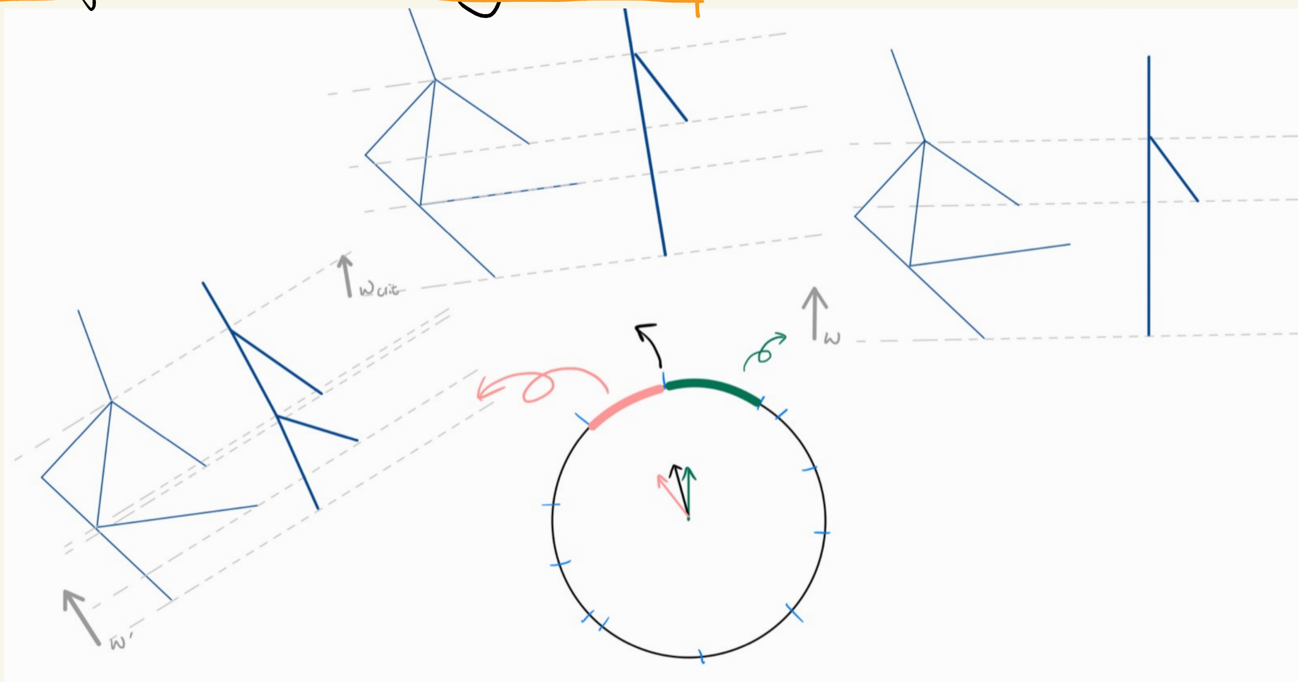
$O(\#\text{simplices} \times 2^{\text{dim of } A} + d)$

$\mathbb{R}^d$

Other variants:

① Merge trees or ~~Reeb graphs~~ ↳ stay tuned!

↳ C, Munch, Percival, Wang 2024



- Not injective in general, but capture some connectivity

↳ good in practice?

② Instead of directions, consider distance to a point, line, or flat:

$\boxed{\text{Onus, Otter, Turkes 2024}}$

Let $PHT(X) : \mathbb{P} \to \mathrm{Dgm}$

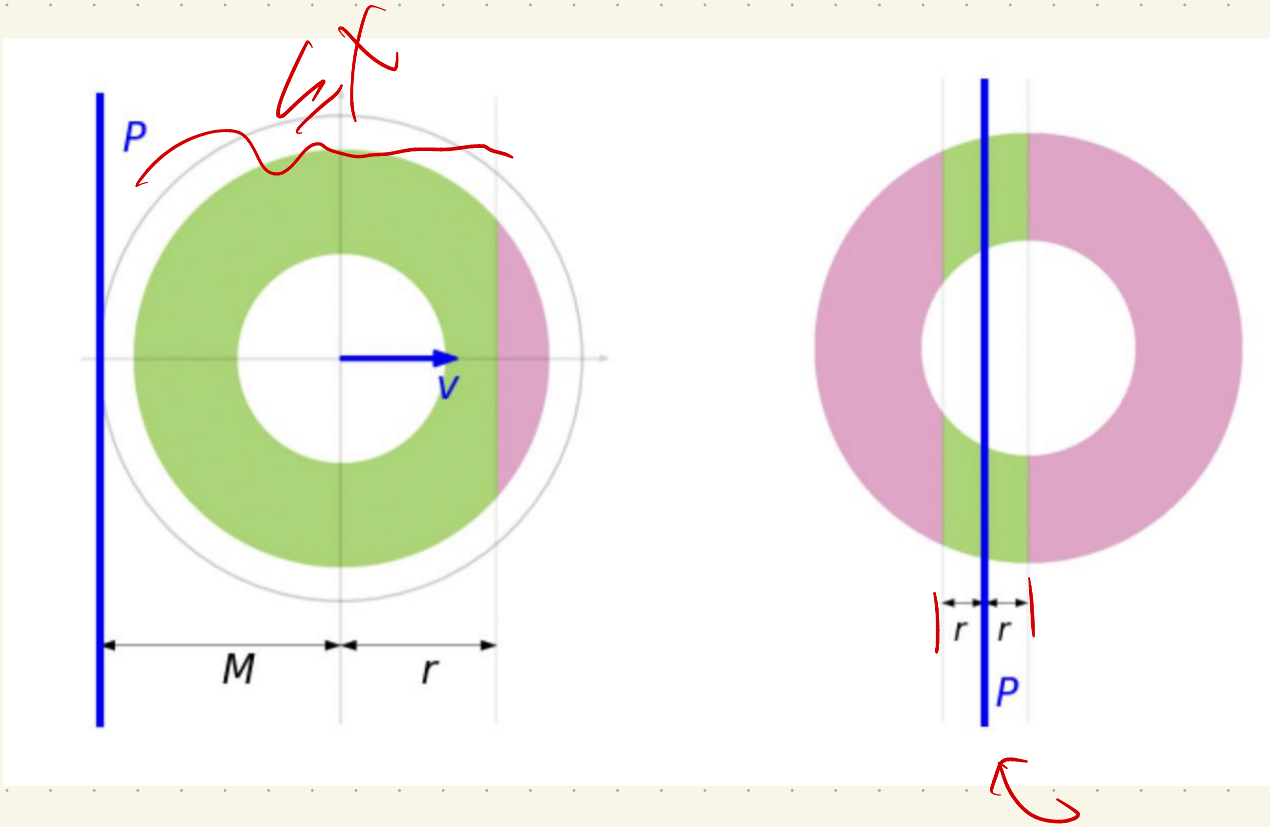
$P \mapsto PD_{f_P}(X)$

where $\mathbb{P}$ is any space

& f_P is distance to P

main example: $\mathbb{P}$ is set of flat subspaces

Why? Fewer directions (we hope)



Any height function is equivalent to distance from a line.

reverse not true: no sublevel filtration

Turkeš, Montúfar + Otter 2022 showed that distance to a line detects curvature + convexity very well

need more complex fp

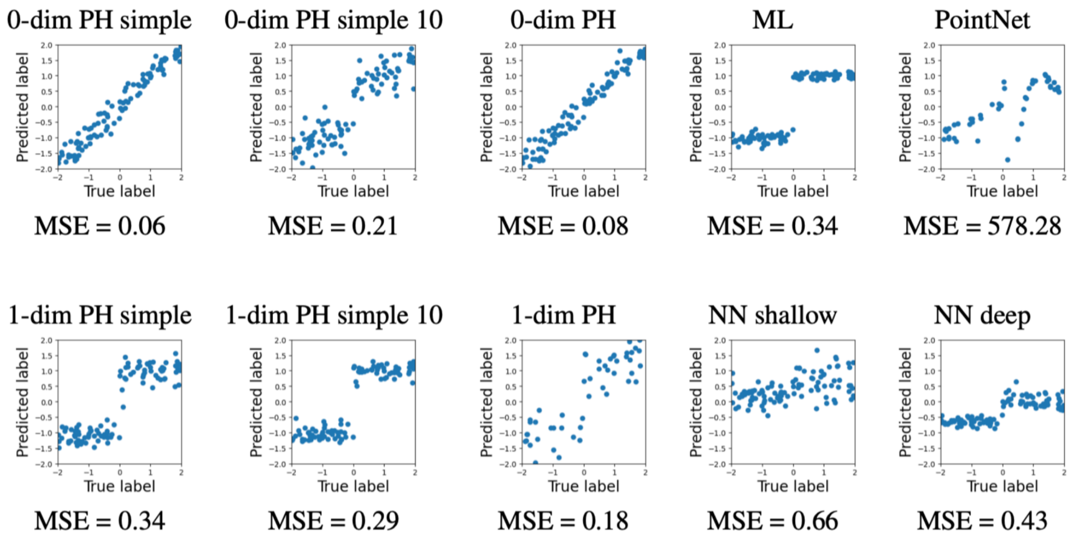


Figure 4: Persistent homology can detect curvature.

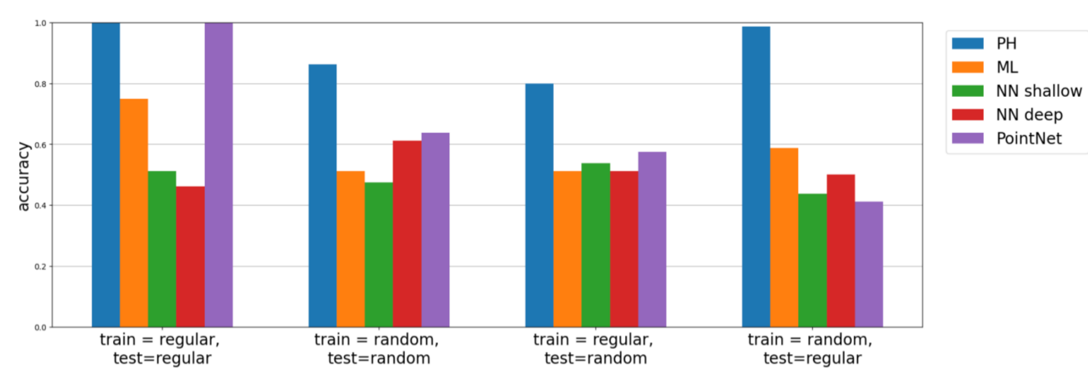


Figure 6: Persistent homology can detect convexity.

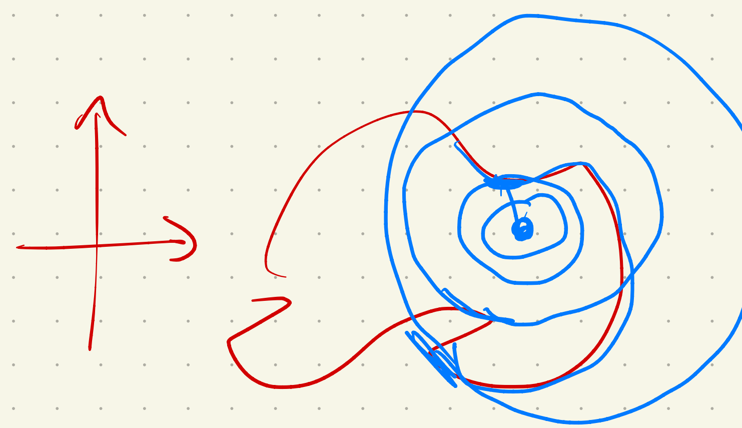
Onus et al 2024 then proved it is injective.

More precisely:

Affine Grassmannians

Name	Notation	$\mathbb{P}$	$\dim(\mathbb{P})$	k
height PHT	$\mathrm{PHT}_{\mathbb{AG}(n-1,n),\mathrm{d}}$	$\mathbb{AG}(n-1,n)$ = hyperplanes in $\mathbb{R}^n$	n	0, 1, . . . , n-2
. . .	. . .	. . .	. . .	. . .
tubular PHT	$\mathrm{PHT}_{\mathbb{AG}(1,n),\mathrm{d}}$	$\mathbb{AG}(1,n)$ = lines in $\mathbb{R}^n$	2(n-1)	0
radial PHT	$\mathrm{PHT}_{\mathbb{AG}(0,n),\mathrm{d}}$	$\mathbb{AG}(0,n)$ = points in $\mathbb{R}^n$	n	“-1”[1]

Prove $\mathrm{PHT}_{AG(m,n),d}$ truncated to homology for degree $0,\dots,m-1$, is injective.



Translating:

$$PHT_{\underbrace{AG(1,n),d}_{\text{lines in } \mathbb{R}^d}}$$

only needs 0-dim PH → completely determine shape

Computational implication:

Filtration	PH Complexity
Distance-to-line	$O(N \log N)$
Height	$O(N^{\omega})$

PH_0

↳ faster Union-Find

↑ matrix alg

Next time:

Looking at some implications of these transforms.