TDA - Fall 2025

Computing Persistence

History Persistence actually came up often!

Matrix algorithm is from

$\boxed{\text{Edelsbrunner-Letscher-Zomorodian 2000}}$

Algebraic formulation given in

$\boxed{\text{Carlsson \& Zomorodian 2004}}$

Independent formulations

$\boxed{\text{Frosni 1990}}$ - manifold comparison in Euclidean space

$\boxed{\text{Robbins 1999}}$ - crystaline structures & periodicity

↳ H_0

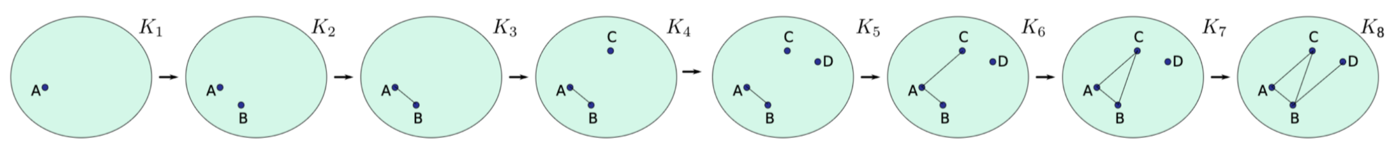
So far: Persistence

Chain complexes & filtrations $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$

$\Downarrow$ pass to homology

$$0 \to H_p(K_1) \xrightarrow{f^{1,2}} H_p(K_2) \xrightarrow{f^{2,3}} \cdots \xrightarrow{f^{n-1,n}} H_p(K_n) \to 0$$

where $f^{i,j}$ is induced by inclusion



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

and $H_p^{i,j} = \operatorname{Im}\left(H_p(K_i) \xrightarrow{f^{i,j}} H_p(K_j)\right)$

↳ homology classes in K_i but checked in bigger complex

Aside: $H_p(K_i)$ is a homology group still!

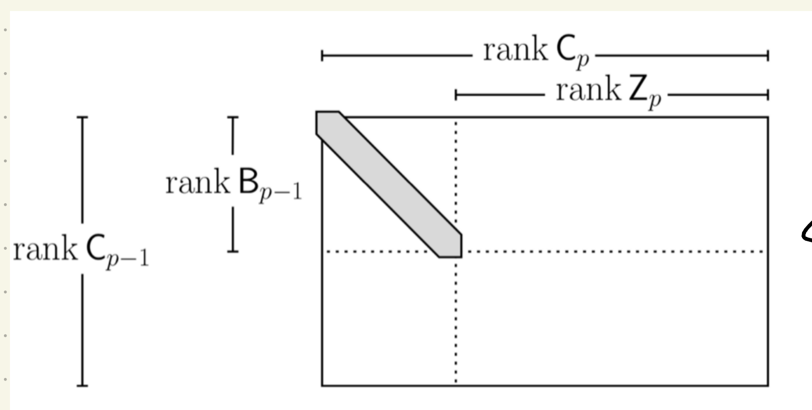
So calculated as we saw before:

$$C_d(K_i) \xrightarrow{\partial_d} C_{d-1}(K_i) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_{p+1}} C_p(K_i) \xrightarrow{\partial_p} \cdots \xrightarrow{\partial_1} C_0(K_i)$$

then $H_p(K_i) = Z_p / B_p$

$= \ker \partial_p / \operatorname{im} \partial_{p-1}$

Calculated via boundary matrix fun :)



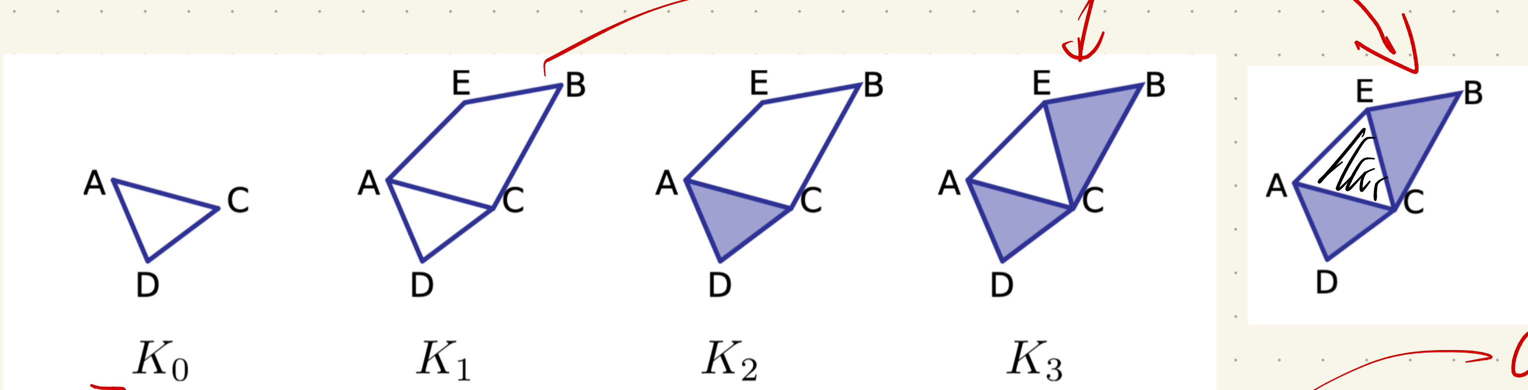
Smith Normal Form

Pairing (book dfn from last class - revisit)

Let $[c]$ be a p^{th} homology class that dies entering X_j. Then, it is born at X_i if + only if $\exists\ i_1 \leq i_2 \leq \cdots \leq i_k = i$ (with $k \geq 1$) s.t.

- $[c_{i_\ell}]$ is born at X_{i_ℓ} $(\ell \in [1..k])$
- $[c] = f_p^{i_1, j-1}([c_{i_1}]) + \cdots + f_p^{i_k, j-1}([c_{i_k}])$
- $i_k = i$ is smallest possible choice

Why??



a_0 $\qquad$ a_4

$$H_1(K_0) \xrightarrow{f_*} H_1(K_1) \xrightarrow{g_*} H_1(K_2) \xrightarrow{h_*} H_1(K_3) \to H_1(K_4)$$

← death! at time = 4 (or in K_4)

[ACE] exists in $H_1(K_3)$, <u>not</u> in $H_1(K_4)$

Birth? could pick [ACE] in K_3

[ABCE] in K_2

[ABCE) in K_1

$f^{1,4}([ABCE])$

Remember: $f: K \to \mathbb{R}$

$$K_1 \subseteq \cdots \subseteq K_i \subseteq \cdots \subseteq K_n$$

induced complex on $f((-\infty, a_i])$

Counting classes & Persistence

Set $\beta_p^{i,j} = \operatorname{rank}(H_p^{i,j})$

∞

$$0 \to H_p(K_1) \to H_p(K_2) \to \cdots \to H_p(K_n) \to 0$$

K_{n+1}

- Attach 0 vector space at end
- Associate $n+1$ to $a_{n+1} = \infty$
- Then $\beta_p^{i,j}$ counts classes born before i which die after j

are alive in j

How can we get # of classes born at i which die at j?

$$H_p^{i-1} \to H_p^{i} \to \cdots \to H_p^{j-1} \to H_p^{j}$$

Pairing function

for $0 < i < j \leq n+1$, define

$$\mu_p^{i,j} = \left(\beta_p^{i,j-1} - \beta_p^{i,j}\right) - \left(\beta_p^{i-1,j-1} - \beta_p^{i-1,j}\right)$$

$\rightarrow$ # of classes born at i that die at j

Why?

$$H_p(X_{i-1}) \xrightarrow{f_p^{i-1,i}} H_p(X_i) \xrightarrow{f_p^{i,j-1}} H_p(X_{j-1}) \xrightarrow{f_p^{j-1,j}} H_p(X_j)$$

$$\underbrace{\quad}_{\beta^{i-1,j-1} - \beta^{i-1,j}} \qquad \qquad \underbrace{\quad}_{\beta^{i,j-1} - \beta^{i,j}}$$

When $\mu_p^{i,j} \neq 0$, the persitence of a class $[c]$, $Per([c])$, which is born at X_i + dies at X_j is defined as $a_j - a_i$.

$\hookrightarrow$ length of barcode

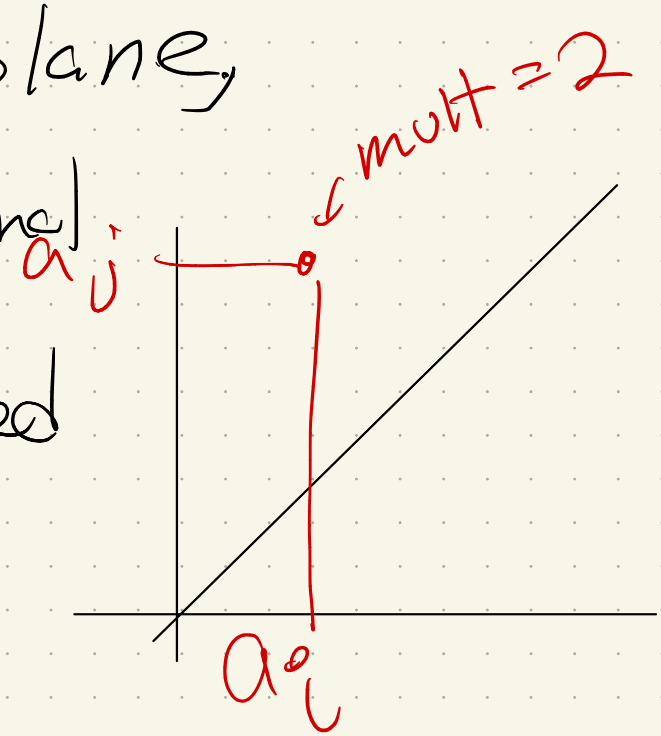
"lifetime"

[If $j = n+1$ with $a_{n+1} = \infty$, $Pers([c]) = \infty$].

Persistence diagram $\mathrm{Dgm}_p(F)$

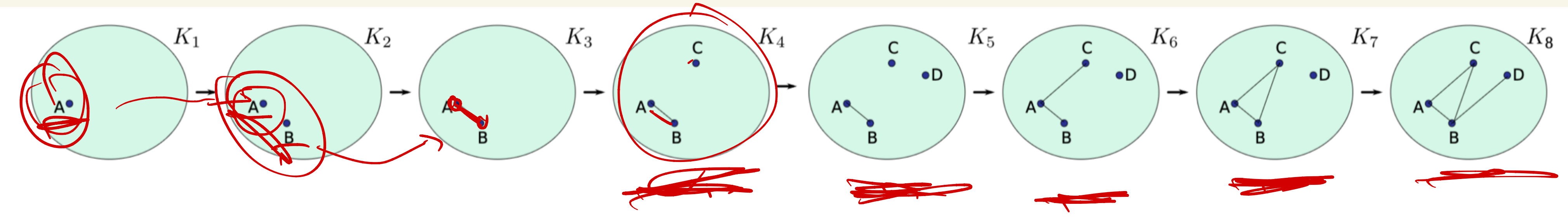
(also written $\mathrm{Dgm}_p(f)$)

Filtration F on K induced by f.
$\mathrm{Dgm}_p(F)$ is obtained by drawing a point (a_i, a_j) with non-zero multiplicity $\mu_p^{i,j}$ $(i<j)$ on extended plane, where points on the diagonal $\Delta = \{(x,x) \in \mathbb{R}^2\}$ are added with infinite multiplicity



Let's try! First, calculate $\beta^{i,j}$

↳ then $\mu^{i,j}$



$$H_0(K_1) \longrightarrow H_0(K_2) \longrightarrow H_0(K_3) \longrightarrow H_0(K_4) \longrightarrow H_0(K_5) \longrightarrow H_0(K_6) \longrightarrow H_0(K_7) \longrightarrow H_0(K_8)$$

$\beta_{i,j}$

j \ i	1	2	3	4	5	6	7	8
∞	0	0	0	0	0	0	0	0
8	1	1	1	1	1	1	1	1
7	1	1	1	1	2	2	2	
6	1	1	1	1	2	2		
5	1	1	1	2	3			
4	1	1	1	2				
3	1	1	1					
2	1	2						
1	1							

$$\mu^{i,j} = (\beta^{i,j-1} - \beta^{i,j}) - (\beta^{i-1,j-1} - \beta^{i-1,j})$$

j \ i	1	2	3	4	5	6	7	8
∞	0	0	0	0	0	0	0	0
8	1	1	1	1	1	1	1	1
7	1	1	1	1	2	2	2	
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4	1	1	1	2				
3	1	1	1					
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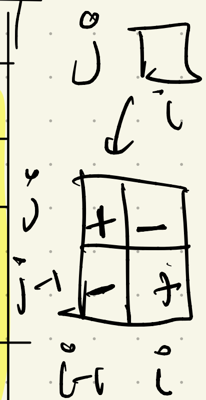
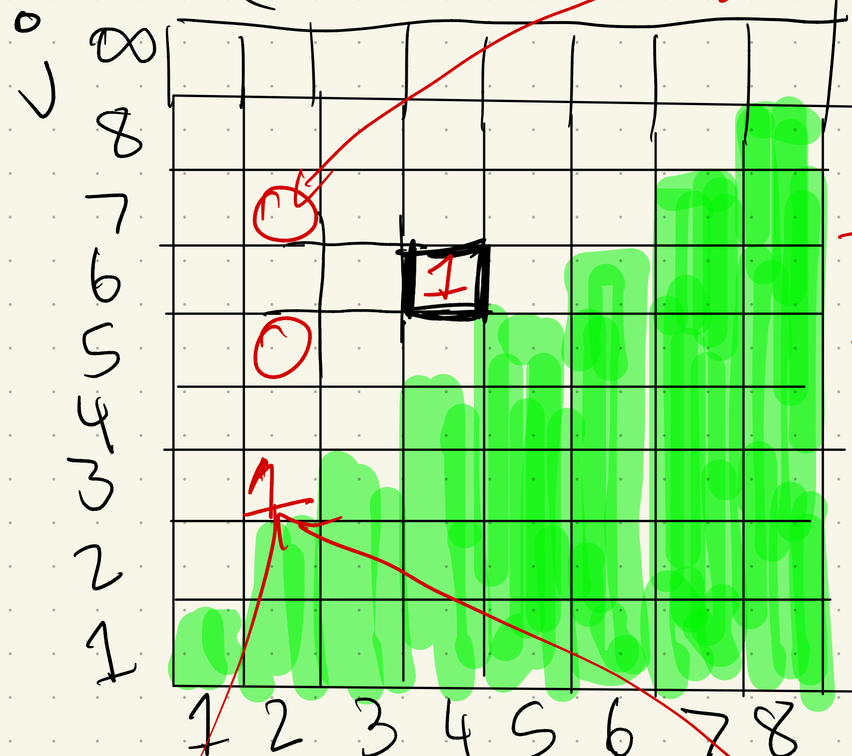
1	1							



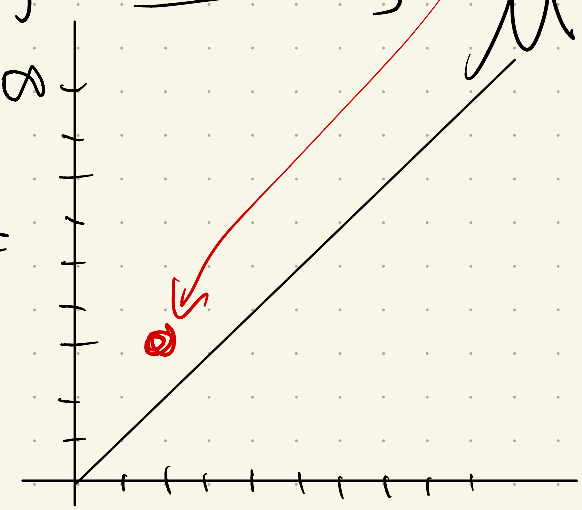


$\mu^{2,7} = \beta^{2,6} - \beta^{2,7} - \beta^{1,6} + \beta^{1,7}$

$\mu^{2,3} = \beta^{2,2} - \beta^{2,3} - \beta^{1,2} + \beta^{1,3}$

$\beta_\infty^{i,j} \longrightarrow \mu^{i,j} \quad (i<j)$

Then $\mathrm{Dgm}(F) =$



Taking stock:

Can compute $H_p(K_i)$.

How to get $H_p^{i,j}$?

Really, want $\beta_p^{i,j}$, so can calculate

$\mu^{i,j} \longrightarrow$ then $Dgm_p(F)$.

So: need to adapt matrix algorithm somehow, to get ranks of induced homology.

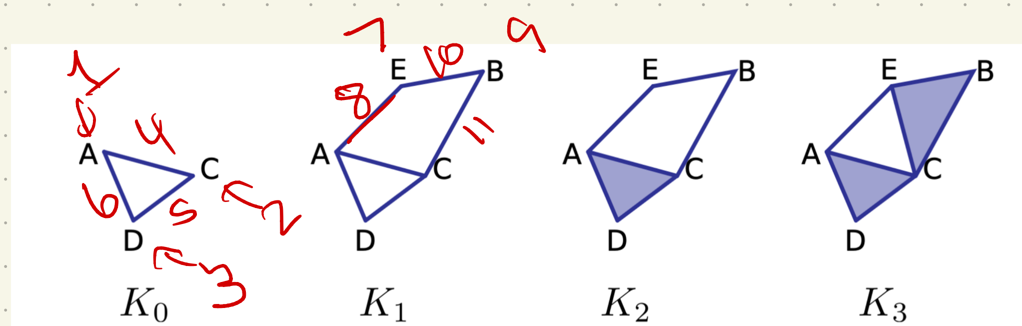
Some Preliminaries

Let $f: K \to \mathbb{N}$ give the index where a simplex σ appears in filtration.

A compatible ordering of the simplices is a sequence $\sigma_1, \sigma_2, \dots, \sigma_m$ s.t.

- $f(\sigma_i) < f(\sigma_j) \Rightarrow i<j$
- $\sigma_i \leq \sigma_i \Rightarrow i<j$

Ex:



Essentially, we now have a simplex-wise filtration: assume $K_j / K_{j-1} = \sigma_j$ is a single simplex.

When p-simplex σ_j is added, two possibilities:

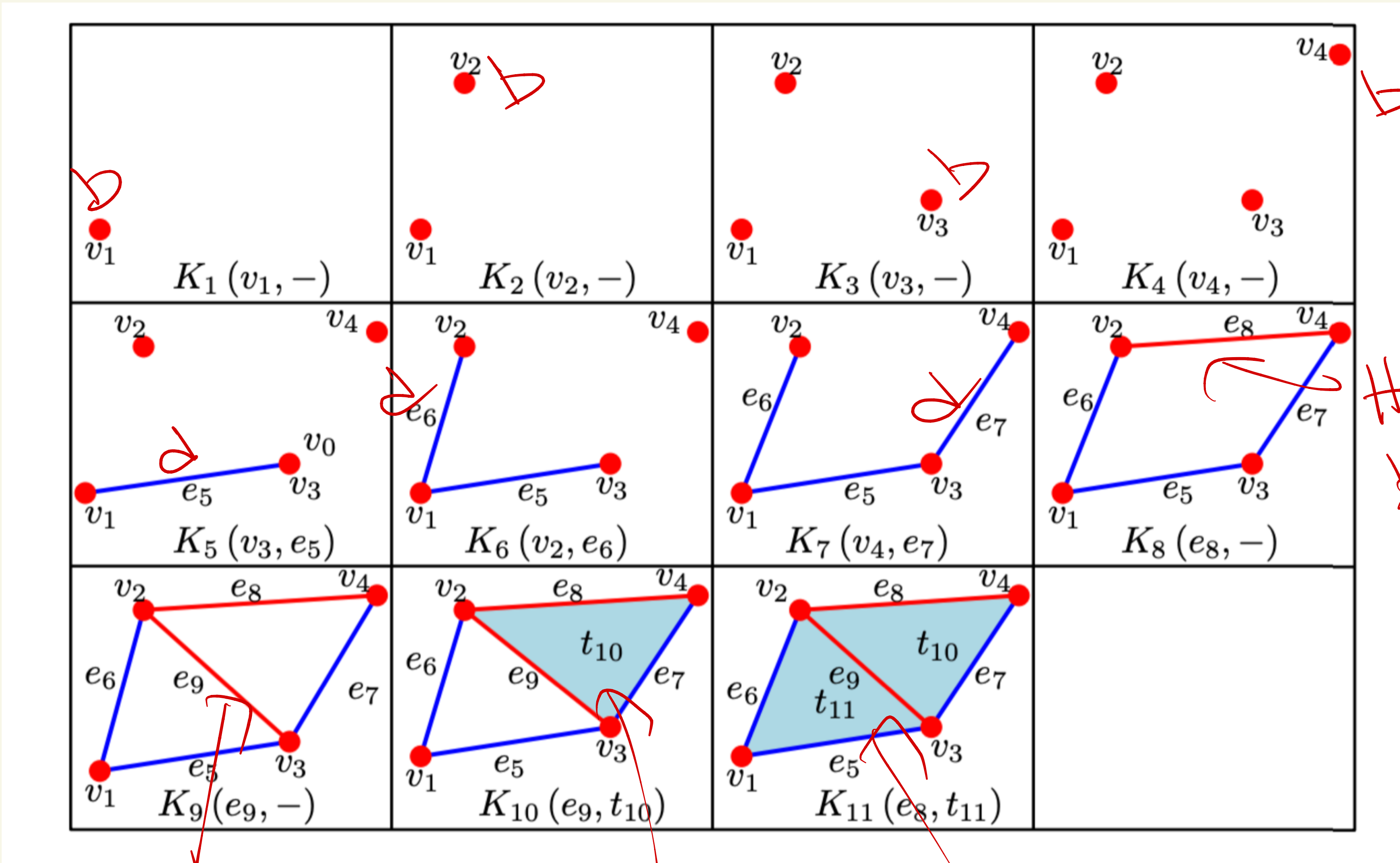
1. A non-boundary p-cycle c along with its classes $[c]+h$ for $h \in H_p(K_{j-1})$ are born. Call σ_j positive (or a creator).

2. An existing $(p-1)$-cycle c along with its class $[c]$ dies. Call σ_j negative (or a destroyer).

Examples

H_0 birth / death
H_1 birth / death

(no H_2 here)



H_1 birth

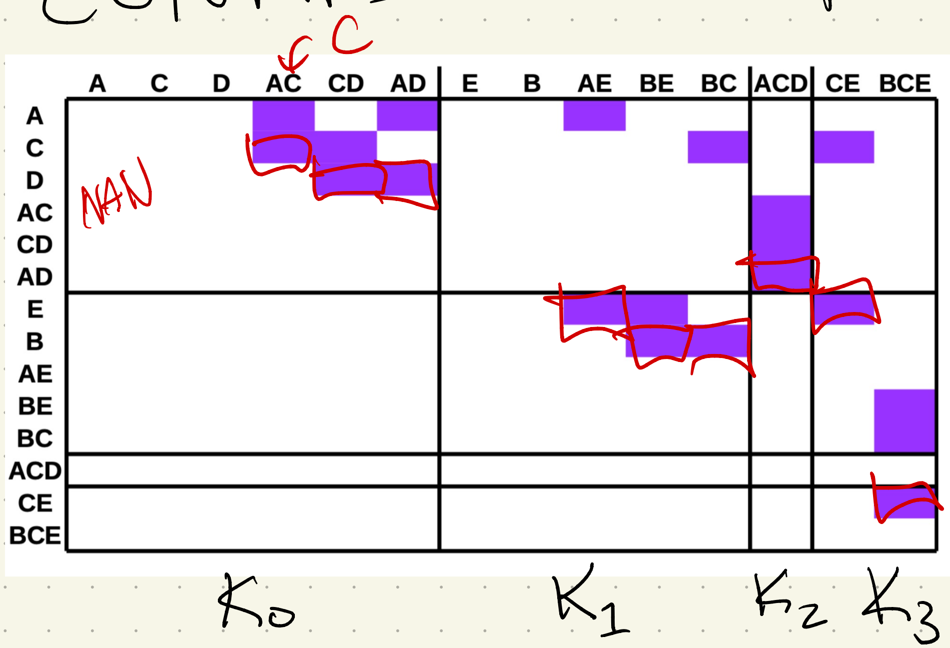
H_1 birth

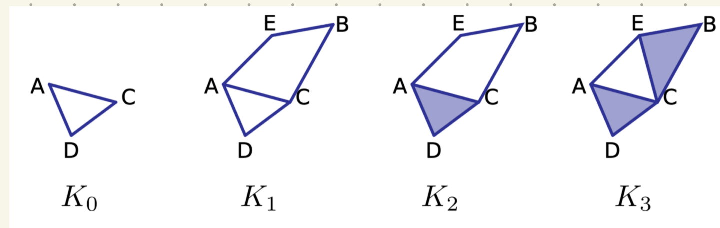
H_1 death

H_1 death

An algorithm

Take boundary matrix, with rows & columns in simplex-wise order:

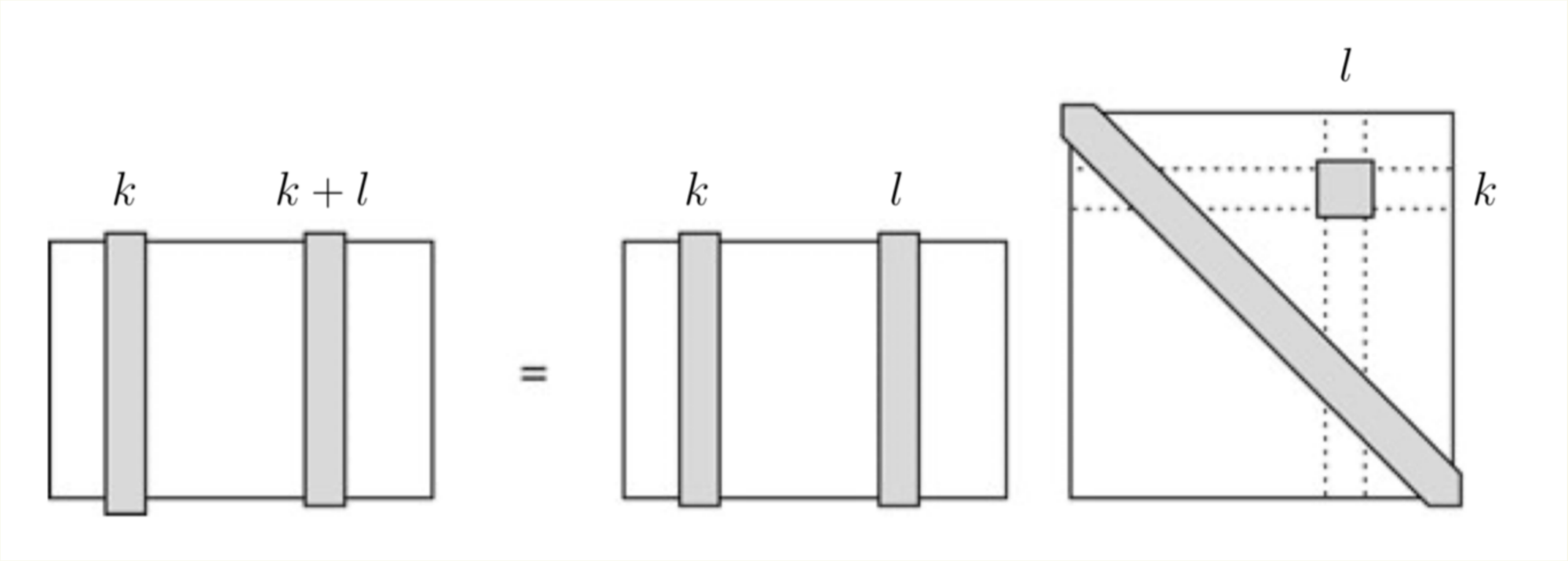




- Let low(j) = row of lowest 1 in column j (& if all 0's, low(j) = NaN)
- R is reduced if $\text{low}(j) \neq \text{low}(j')$ for any $j \neq j'$

Matrix operations

To add row k to row l, can create matrix with 1 in l, k:



Here!

$R = B$

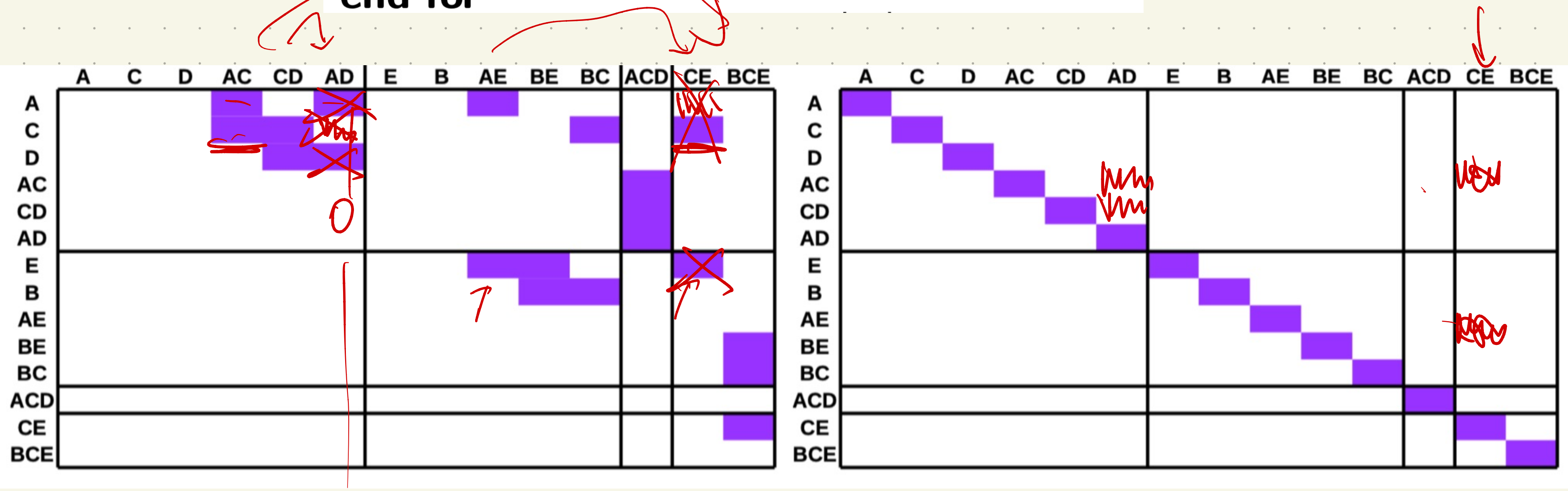
for $j = 1 \cdots m$ **do**

  while $\exists\, j' < j$ with $low(j') = low(j)$ **do**

    add column j' to column j

  end while

end for



Idea

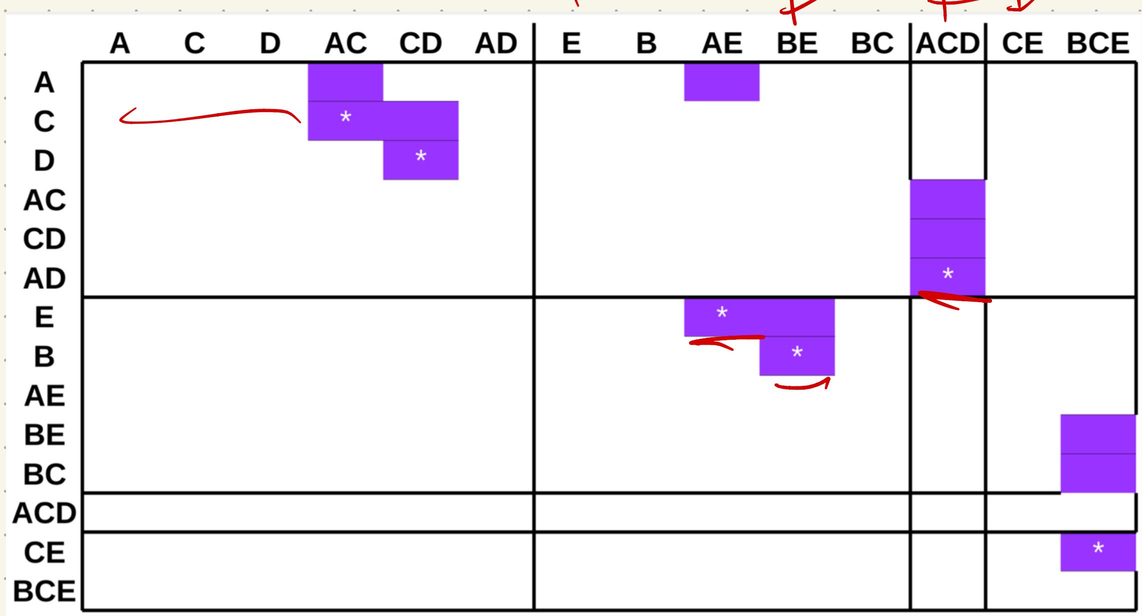
- B is upper triangular, & if we add from left it stays that way
- If a column is entirely 0, that simplex created a homology class (so it is positive)
- If a column has a lowest 1, then this simplex killed a class from the previous step.

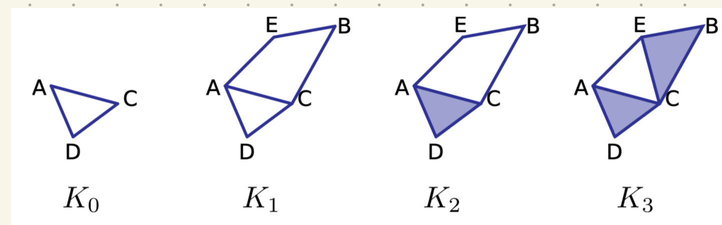
unique (& indep of operations)

[Proof: essentially, 1 in that spot means $\mu^{i,j} = 1$]

Pairing

- Every negative simplex must be paired with a previous positive (birth/death)
  - ↳ pair with its lowest 1





Pairs:

AC → C

CD → D

AE → E

BE → B

ACD → AD

BCE → CE

Fact

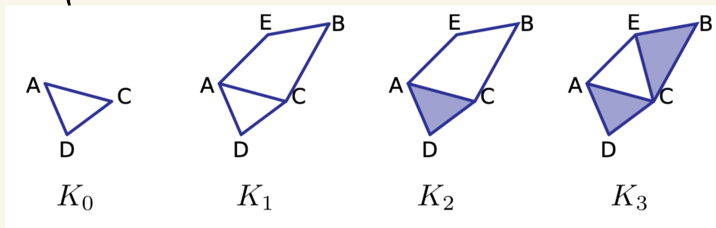
The number of unpaired p-simplices in a simplex-wise filtration of K is its p^{th} Betti number.

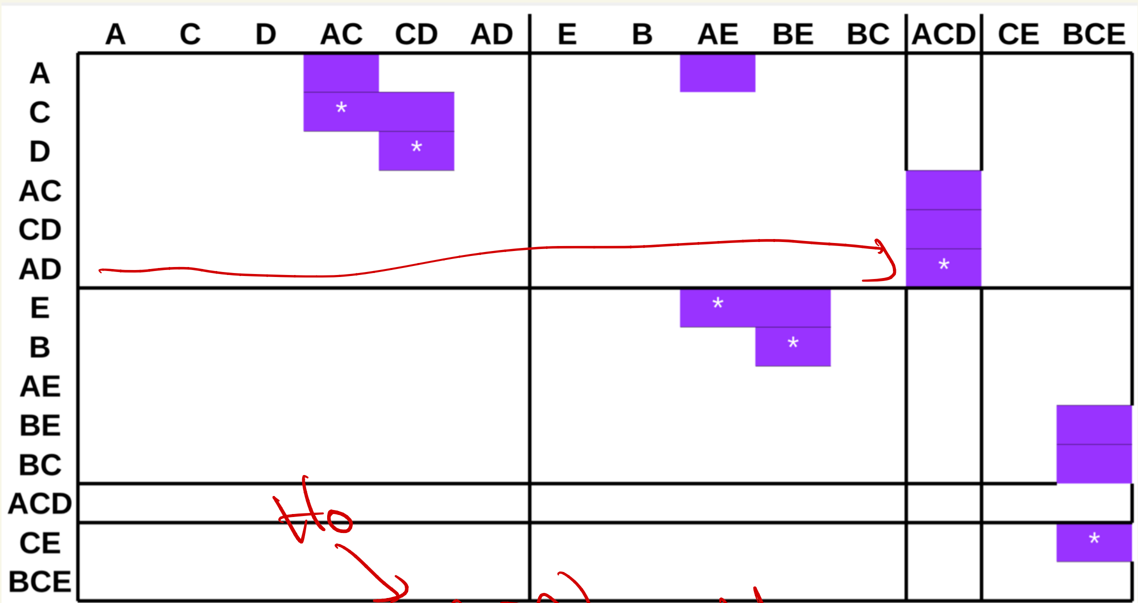
Why?

If paired, birth $\leftrightarrow$ death

If unpaired must have created an H_p feature

So: use pairs to build persistence diagram.





Unpaired:

K_0: A and AD

$\beta_0(K_0) = 1$ + $\beta_1(K_0) = 1$

K_1: still A + AD,

plus BC

$\beta_0 = 1$ $\beta_1 = 2$

K_2: AD now paired!

$\beta_0 = 1$ $\beta_1 = 1$

K_3: no change

Next time:

Some code discussion

Stability + distance metrics for persistence

Longer term:

- Statistics + ML
- Reeb graphs + Mapper graphs
- Extensions of persistence