CSE/AMCS 60973: Topological Data Analysis
Homework 2

This assignment is due on Sept 22, by 11:59pm, and covers the material in chapter 2 of our textbook by Dey and Wang.

As with Homework 1, you are welcome to consult any sources or work in groups. But, the same rules apply, sp please be sure to:

- Include an acknowledgement section, listing any students you worked with or other resources you used - including chatgpt!
- Type up your own solution. I encourage you to use latex, since it's a necessary tool in our field.
- Make sure you type up your own solutions, as verbatim copying will be given a 0.

Required Problems

Complete any 3 of the following problems.

Note: For the last 3 questions, don't use existing TDA packages. Stick to standard tools like numpy, networkx, scipy, matplotlib, etc. - I'm flexible on language, although please touch base with me to let me know which you're planning (an email is fine). It's highly probable your code will be slower than the available TDA tools - that's fine! The idea is to get a good sense of what goes into computing these things. Turn in your work for this portion as a pdf export of your code, annotated or commented appropriately to show the computations. For example, you can submit a pdf export of a well-annotated jupyter notebook, or a session of using the code at the prompt with matrices and operations shown, or even the output of the executable showing the computation that results from your program somehow.

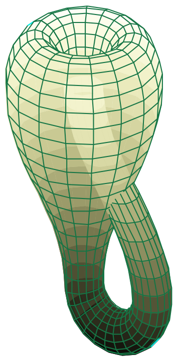
1. Let K be a geometric simplicial complex. Prove that K is a geometric realization of the nerve of the collection of vertex stars in K.

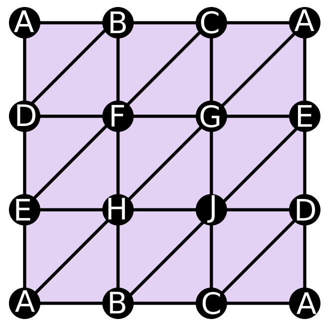
2. Suppose we have a collection of sets $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ where there exists an element $U \in \mathcal{U}$ that contains all other elements in $\mathcal{U}$. Prove that the nerve complex $N(\mathcal{U})$ is contractible to a point.

3. Let K be a triangulation of a 2-dimensional sphere S^2. Now remove h vertex disjoint triangles from K, and let the resulting simplicial complex be K'. Describe the Betti numbers of K', and justify your answer.

4. Give an example where a simplex which is weakly witnessed may not have all its faces weakly witnessed. Prove that (i) $W(Q, P') \subseteq W(Q, P)$ for $P' \subseteq P$, and (ii) $W(Q', P)$ may not be a subcomplex of $W(Q, P)$ when $Q' \subseteq Q$.

5. We can represent the Klein bottle (visualized on the left below) by the following simplicial complex (right):





Note the repeated letters on vertices on the boundaries: these should be thought of as the same vertex, just drawn in two places so we can see this complex flat.

Do the following (fixing $\mathbb{Z}_2$ homology throughout):

(a) Determine β_0, β_1, and β_2 for the Klein bottle.

(b) Compute generators for each of the homology groups $H_i(K)$, $i = 0, 1, 2$.

(c) Describe two other topological spaces (simplicial complexes) that have the same Betti numbers when using $\mathbb{Z}_2$ coefficients, but are not homotopy equivalent.

You can solve this problem with the aid of code, in which case you should explain your reasoning and give enough snippets of code for me to understand how you did your calculation. If you'd prefer to determine everything by hand, that is also fine, but be sure to include justification for your answer.

6. Write code for the function `RipsGraph(P,d)` as follows. (I suggest using networkx for the graph part, but any graph library is fine if you prefer others.)

- Inputs:
  - P - a set of points $P \subset \mathbb{R}^2$ given as a $k \times 2$ numpy matrix
  - d - a diameter $d \geq 0$
- Output:
  - a graph which is the 1-skeleton of the Rips complex $VR(P, d)$.

Create a point set of 15 random points in the box $[0, 1] \times [0, 1]$. Draw this graph as an overlay on the points (even though there will likely be crossings).

7. Write code for the function `RipsComplex(P,d)`.

- Inputs:
  - P - a set of points $P \subset \mathbb{R}^2$ given as a $k \times 2$ numpy matrix
  - d - a diameter $d \geq 0$
- Output:
  - The boundary matrix representing the 3-skeleton of $VR(P, d)$.