Recap

- If you are not turning in HW9 today
  ↳ by Friday!!
- Worksheet is posted
- Practice Final - bring on Friday
- This Friday: set review session
- Don't forget: evaluations!

Reading:

Intro to LP.
(Really, just motivating why we should solve these.)

Why? Everywhere!

Mainly setup so far
↳ algorithm is coming!

The algorithm: Simplex

Assumes canonical form:

$$\text{maximize} \sum_{j=1}^{d} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for each } i = 1 \mathinner{..} n$$

$$x_j \geq 0 \quad \text{for each } j = 1 \mathinner{..} d$$

So:

- no min
- only $\leq$
- $x \geq 0$ for all variables

↳ slack variables

How to create canonical form?

① Turn max to min:

multiply by -1

More specifically:

$$\min\ c_1x_1 + c_2x_2 + \cdots + c_dx_d$$

$$\Longleftrightarrow \max\ -c_1x_1 - c_2x_2 \cdots - c_dx_d$$

② for $\geq$:

$$\sum_j a_{ij}x_j \geq b_i$$

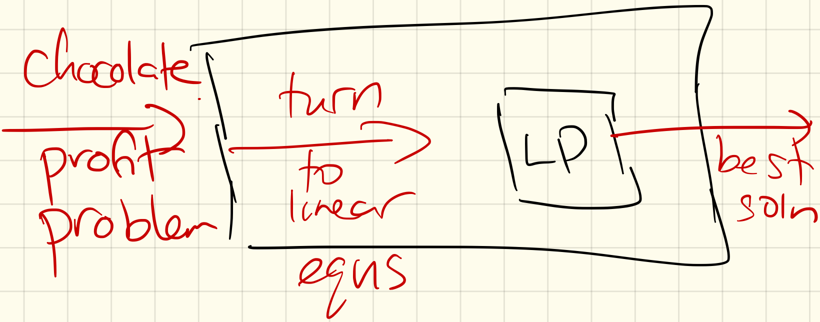
$$\Longleftrightarrow -1(\qquad\qquad)$$

$$= \sum_j -a_{ij}x_j \leq -b_j$$

Connections to other problems:

It turns out that LPs are powerful enough to express <u>many</u> types of problems.

In a sense, we solve many problems by <u>reducing</u> them to an LP:



Ex: Flows & Cuts

Input: directed G w/ edge capacities $c(e)$ & $s, t \in V$

Goal: Compute flow $f: E \to \mathbb{R}$

s.t.

① $0 \leq f(e) \leq c(e)$

② $\forall v \neq s, t,$

$$\sum_u f(u \to v) = \sum_w f(v \to w)$$

Make an LP:

Maximize: $\displaystyle\sum_{e \text{ out of } s} f(e)$

s.t. for each e, $f(e) \geq 0$

$f(e) \leq c(e)$

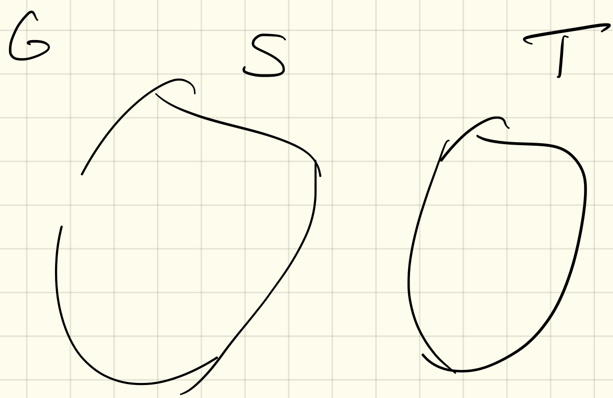
for each vertex v: $\displaystyle\sum_u f(u \to v) = \sum_w f(v \to w)$

Related: Min cuts (S,T)

Use <u>indicator variables</u>:

$$S_v = 0 \text{ or } 1$$

$$X_{u \to v} = 1 \text{ if } u \in S \text{ and } v \in T$$



The LP: Min cut

$$\text{Minimize} \quad \underbrace{\sum_{u \to v} c_{u\to v} \cdot x_{u\to v}}_{\text{cost of cut}}$$

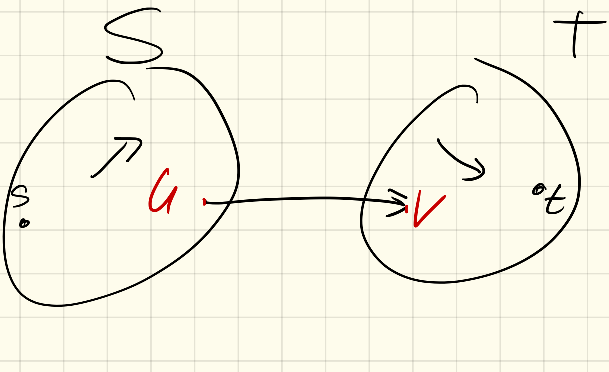
s.t.

$$x_{u\to v} + s_v - s_u \geq 0 \quad \forall u, v$$

$$x_{u\to v} \geq 0 \quad \forall u, v$$

$$s_s = 1$$

$$s_t = 0$$



Note:

For that example, a solution to flow/cuts would yield optimal LP solution.

The reverse is not obvious!

LP might have strange fractional answer which doesn't describe a cut.

It can be shown that this won't happen

↳ but not obvious...

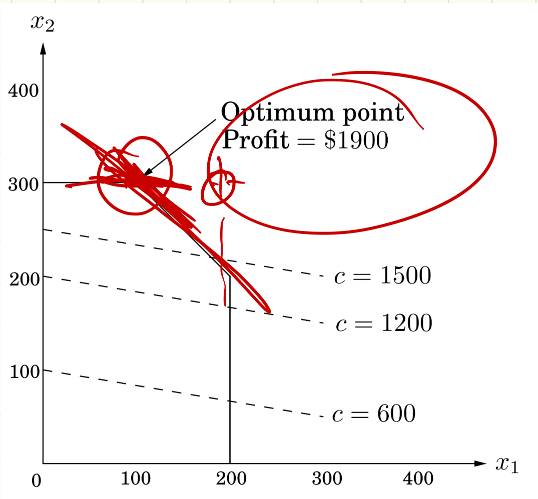
Duality:

Recall our chocolate:

LP: $\max\ x_1 + 6x_2$

s.t.

$$x_1 \le 200$$
$$x_2 \le 300$$
$$x_1 + x_2 \le 400$$
$$x_1, x_2 \ge 0$$



Can we check that this is best?

$$\max \quad \overset{1}{x_1} + \overset{6}{6 x_2}$$

s.t.

$x_1 \le 200$ ① ← 1

$x_2 \le 300$ ② ← 6

$x_1 + x_2 \le 400$

$x_1, x_2 \ge 0$

Play w/ inequalities:

① + 6 · ②:

$$1(x_1) + 6(x_2) \le 200 \cdot 1 + 300 \cdot 6$$

$$x_1 + 6x_2 \le 2000$$

Profit must be ≤ 2000

Interesting!

These 2 inequalities tell us that we couldn't ever beat \$2000.

But recall soln was \$1900 — can we get a better combo?

$$\max \quad x_1 + 6x_2$$

s.t.

$$x_1 \le 200 \qquad (1) \quad : 0$$
$$x_2 \le 300 \qquad (2) \quad : 5$$
$$x_1 + x_2 \le 400 \qquad (3) \quad : 1$$
$$x_1, x_2 \ge 0$$

magic coeffs

Play: $0 \cdot (1) + 5 (2) + 1 (3)$

$$5x_2 + x_1 + x_2$$
$$\le 5 \cdot 300 + 400$$

$$\Rightarrow x_1 + 6x_2 \le 1900$$

These multipliers, $(0, 5, 1)$, are a <u>certificate of optimality</u>.

↳ No valid solution can ever beat \$1900

But how do we find these magic values??

In this, we had three "$\leq$" inequalities

↳ So goal is to find the right 3 multipliers:

y_1, y_2, and y_3

Duality!

Multiplier		Inequality
y_1	$\times$	$x_1 \le 200$
y_2	$\times$	$x_2 \le 300$
y_3	$\times$	$x_1 + x_2 \le 400$

Result

$$y_1 (x_1 \le 200)$$
$$y_2 (x_2 \le 300)$$
$$y_3 (x_1 + x_2) \le 400$$

Note: Make left side look like the original max/min goal, so right will be an upper bound

$$y_1 x_1 + y_2 x_2 + y_3 x_1 + y_3 x_2 \le y_1 200 + y_2 300 + y_3 400$$

So here:

$$(y_1 + y_3) x_1 + (y_2 + y_3) x_2 \leq 200 y_1 + 300 y_2 + 400 y_3$$

Means:

$$x_1 + 6 x_2 \leq 200 y_1 + 300 y_2 + 400 y_3$$

if: $\begin{cases} y_1, y_2, y_3 \geq 0 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \end{cases}$ $x_1 + 6x_2$

Any y_i's would give an upper bound!

We want the best one

↳ ie minimize another LP!

Duality:

$$\max \quad x_1 + 6x_2$$
s.t.
$$x_1 \le 200 \qquad \times y_1$$
$$x_2 \le 300 \qquad \times y_2$$
$$x_1 + x_2 \le 400 \qquad \times y_3$$
$$x_1, x_2 \ge 0$$

$\Updownarrow$ Dual

$$\min \quad 200y_1 + 300y_2 + 400y_3$$
s.t.
$$y_1 + y_3 \ge 1$$
$$y_2 + y_3 \ge 6$$
$$y_1, y_2, y_3 \ge 0$$

Any solution to bottom is upper bnd to top LP.

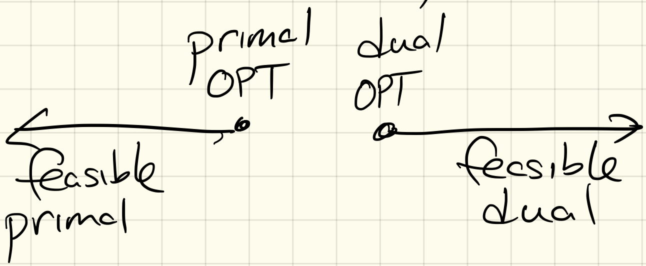
$\Longrightarrow$ If we can find primal/duals that are equal, both are OPT

Here, 1900: primal $(x_1, x_2) = (100, 300)$

Dual: $(y_1, y_2, y_3) = (0, 5, 1)$

This is just like max flow/min cut duality, in a way.

Works for any LP!



this gap – the duality gap – is = 0.

In general:

Primal LP	Dual LP
$\max\ c^T x$	$\min\ y^T b$
s.t.	s.t.
$Ax \leq b$	$y^T A \geq c^T$
$x \geq 0$	$y \geq 0$

Recall our chocolate:

$\max\ x_1 + 6x_2$	$\min\ 200y_1 + 300y_2 + 400y_3$
s.t.	s.t.
$0x_2 + x_1 \leq 200$	$y_1 + y_3 \geq 1$
$0x_1 + x_2 \leq 300$	$y_2 + y_3 \geq 6$
$x_1 + x_2 \leq 400$	$y_1, y_2, y_3 \geq 0$
$x_1, x_2 \geq 0$	

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad [y_1\ y_2\ y_3]\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$