Algorithms

Max flow / min cut theorem

Today

- Reading due Wed. (Ch 11) (bigger than usual!)
- Oral grading next Monday + Tuesday
- Sub on Wed + Fri.
- Rest of Chapter 10 - next week

More formally:

Given a directed graph with two designated vertices, s and t.

Each edge is given a capacity c(e).

Assume:

- ~~No edges enter s.~~
- ~~No edges leave t.~~
- Every $c(e) \in \mathbb{Z}$.

integer capacity

Goal:

Max flow: find the most we can ship from s to t without exceeding any capacity

Min cut: find smallest set of edges to delete in order to disconnect s + t

Flows:

A <u>flow</u> is a function $f: E \to \mathbb{R}^+$, where $f(e)$ is the amount of flow going over edge e.

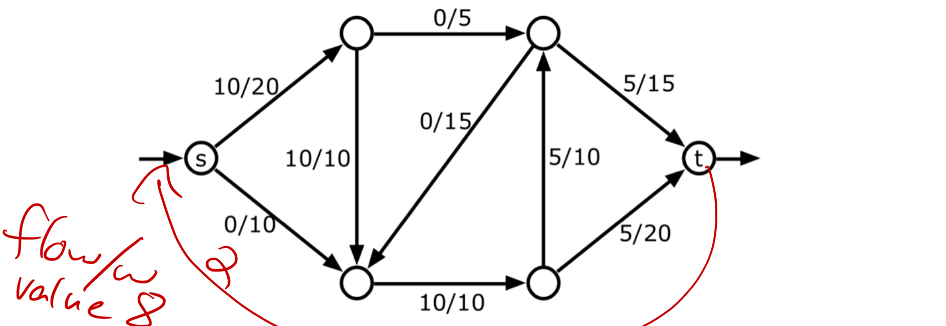
Must satisfy 2 things:

- Edge constraints:

  $$0 \le f(e) \le c(e)$$

- Vertex constraints:

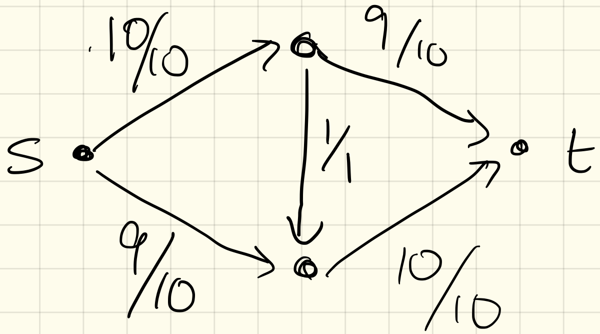
for $v \ne s$ or t: $$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$



An (s,t)-flow with value 10. Each edge is labeled with its flow/capacity.

$$\text{Value}(f) = \sum_{e \text{ out of } s} f(e) - \sum_{e \text{ into } s} f(e)$$

Why can't we just be greedy?



Are there any more flow paths?

Yes - but need to "unflow" to find it.

Cuts:

An s-t cut is a partition of the vertices into 2 sets, S and T, so that:

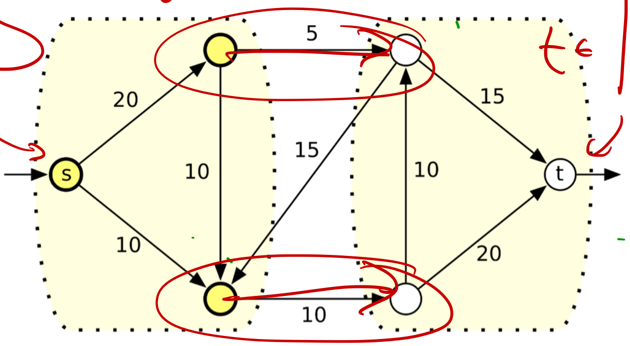
- $s \in S$
- $t \in T$
- $S \cap T = \emptyset$, $S \cup T = V$

The capacity of a cut is

$$\sum_{\substack{\vec{uv} \in E \\ \text{with } u \in S,\, v \in T}} c(\vec{uv}) = \|S, T\|$$

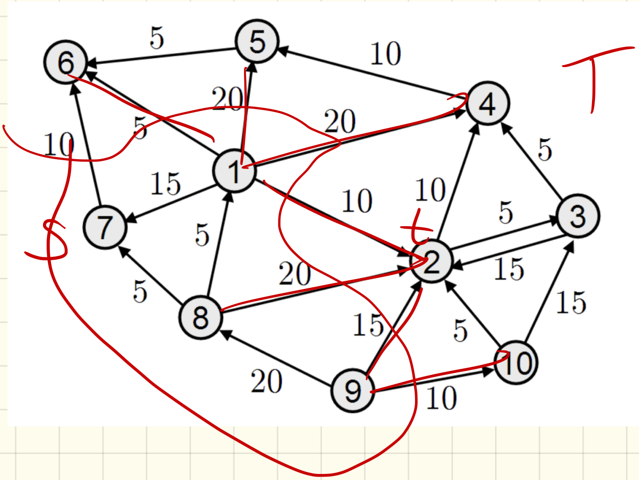
edges out of S

$s \in S$ $\quad$ $t \in T$



An (s, t)-cut with capacity 15. Each edge is labeled with its capacity.

Cuts: not always so obvious!



S

large cut!

Thm: (Ford-Fulkerson '54, Elias-Feinstein-Shannon '56)

The max flow value

$= $ min cut value

One way is easy:

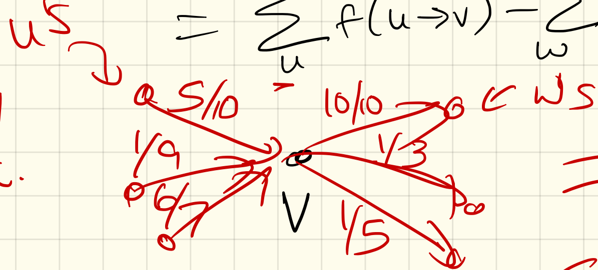
Any flow $\leq$ any cut.

Pf: Pick a flow f:

Let $\delta f(v) =$ flow out of vertex v

$= \sum_u f(u \to v) - \sum_w f(v \to w)$

for any $v \neq s,t$.

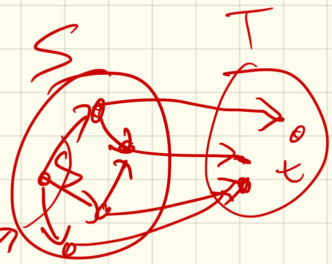


$= 0$

$|f| = \sum_{v \in S} \partial(v)$

cut

Why?



$|f|$

Next:

$$|f| = \sum_{v\in S}\Bigg[\underbrace{\sum_u f(u\to v) - \sum_w f(v\to w)}_{\delta f(v)}\Bigg]$$

$$= \sum_{v\in S}\sum_u f(u\to v) - \sum_{v\in S}\sum_w f(v\to w)$$

Then, can remove any $S\to S$ edges, so only $S\to T$ edges left:

$$= \sum_{v\in S}\sum_{\substack{\cancel{w\in S}\\ w\in T}} f(v\to w) - \sum_{v\in S}\sum_{\substack{\cancel{w\in S}\\ w\in T}} f(w\to v)$$

$$\leq \sum_{v\in S}\sum_{w\in T} \underbrace{f(v\to w)}_{\leq c(v\to w)}$$

$$\leq \sum_{v\in S}\sum_{w\in T} c(v\to w)$$

$$= \|S,T\|$$

Next: Show that can get them equal:

Key tool in proof:

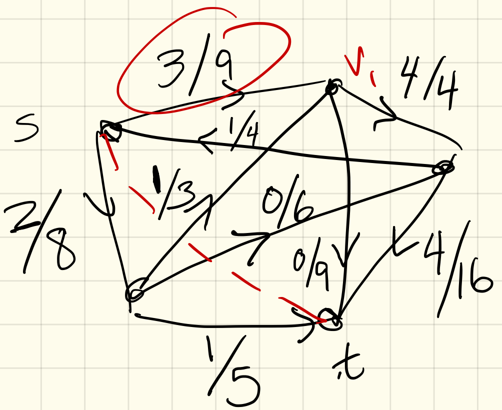
Residual capacity: Given G & F:

$$C_f(u \to v) := \begin{cases} c(u\to v) - f(u \to v) & \text{if } u \to v \in E \\ f(u \to v) & \text{if } v \to u \in E \\ 0 & \text{otherwise} \end{cases}$$

Ex:

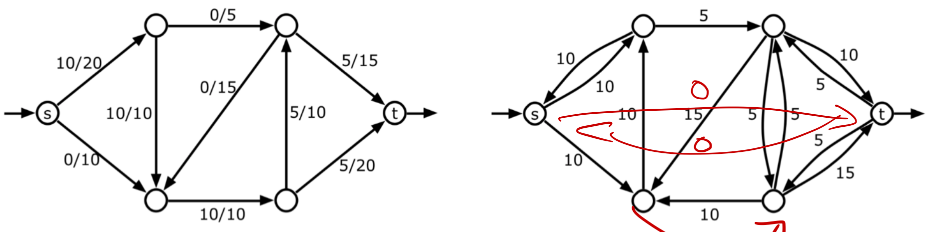
$$S \xrightarrow{3/9} v_1$$

$$C_f(S \to v_1) = 9 - 3 = 6$$



$$C_f(v_1 \to S) = 3$$

We can visualize this as a new graph, G_f:

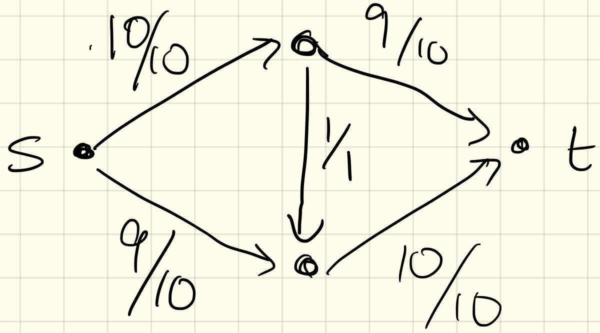


A flow f in a weighted graph G and the corresponding residual graph G_f.

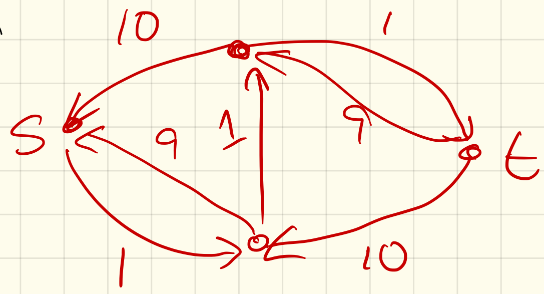
Intution:

A path in G_f if a way to send more flow!

Another example:

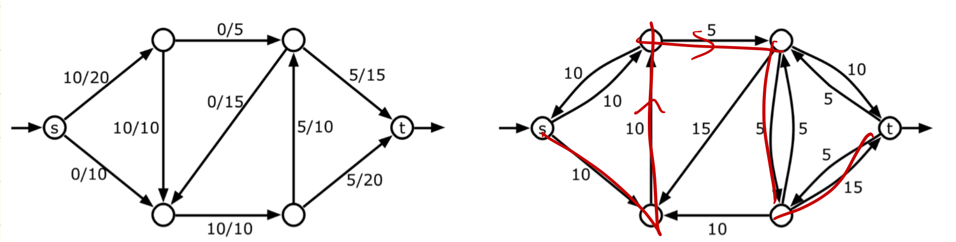


G_f:



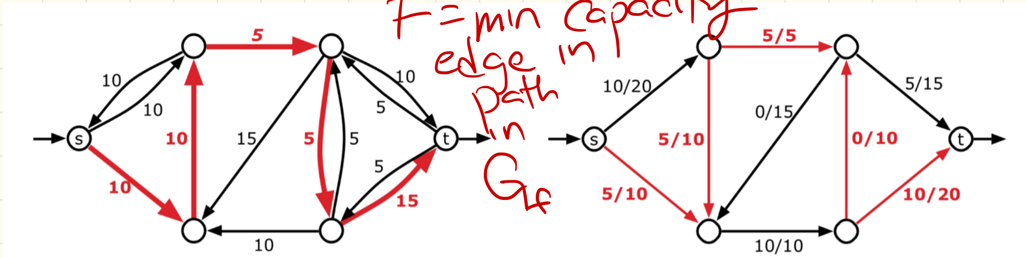
Augmenting a path:

Suppose there is a path in G_f from s to t:



A flow f in a weighted graph G and the corresponding residual graph G_f.

F = min capacity edge in path in G_f



An augmenting path in G_f with value $F = 5$ and the augmented flow f'.

$f' = \begin{cases} \text{if } u \to v \text{ not on aug. path, } f'(u\to v) \text{ is } f(u \to v) \\ \text{if } u \to v \text{ is in } G, \; f'(u \to v) = f(u \to v) + F \\ \text{otherwise } (u \to v \text{ is not in } G) \; f'(v \to u) = f(v \to u) - F \end{cases}$

Claim: f' is also a feasible flow!

Why?

- For any $u \to v$ not on augmenting path,

  same flow value

- For $u \to v$ on augmenting path,

$$f'(u \to v) = f(u \to v) + F$$
$$\geq f(u \to v) \geq 0$$

Still feasible!

unpush

more

So: f wasn't a max flow, since f' is larger.

On other hand:

if G_f has no $s \to t$ path, find $|S|$ = set of vertices that s can reach.

Claim: $(S, V-S)$ is a cut.

(+ f uses every $S \to V-S$ edge to its max capacity)