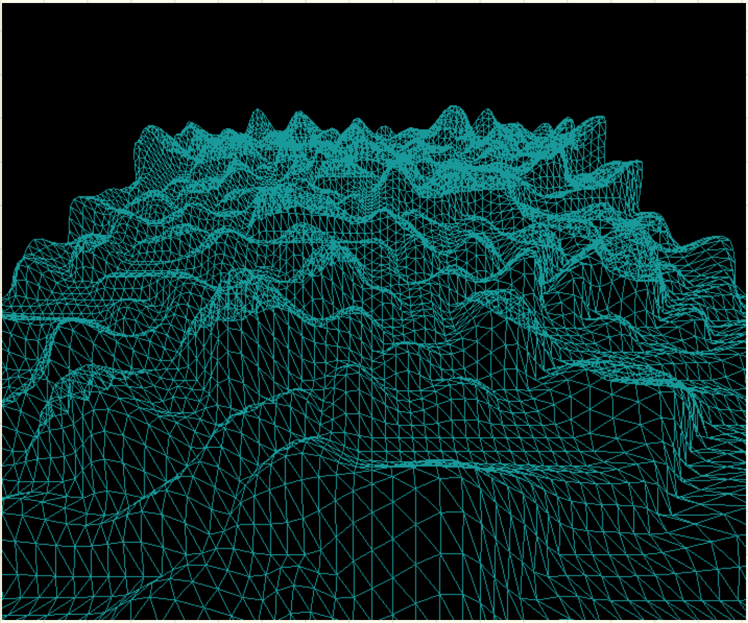
Computational Geometry

Tools + constructs for comp. topology

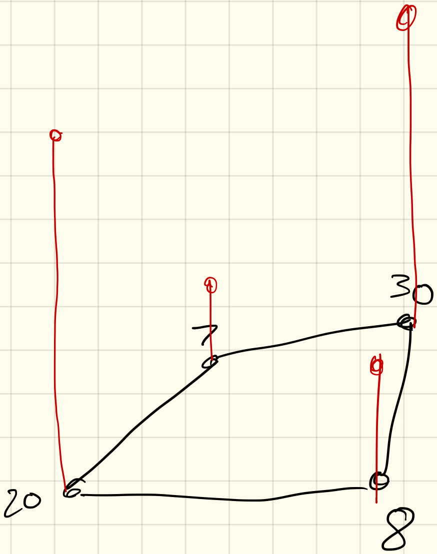
Concept: "Nice" triangulations

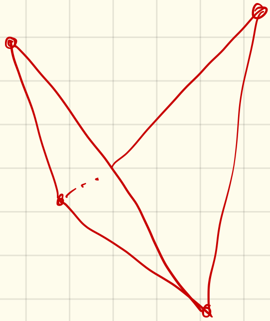
Consider a terrain: a 2-d simplicial complex where each vertex gets a height:

Question: Which is best?



The actual triangles are key:

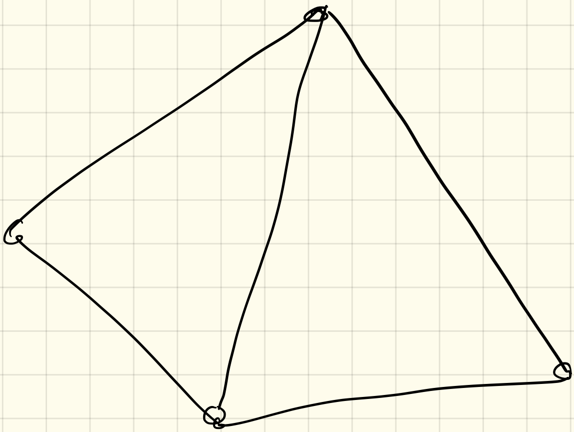


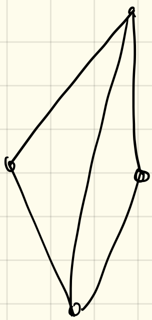


vs

Intuition:

In the plane, with no 4 points co-circular, consider 2 adjacent triangles.

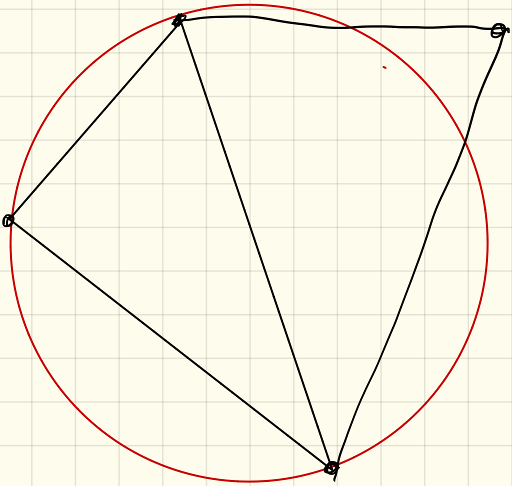




Delaunay triangulation:

A triangulation is Delaunay if no point x is in the interior of any triangles circumcircle.

↳ circumsphere



Voronoi diagrams

- Old & fundamental concept
  - Gregory Voronoi in 1908
  - also attributed to Drichlet
  - but re-invented and used by physicists, meteorologists, ...

Definition:
Given a set P of sites, the Voronoi diagram is a subdivision of space into the cells

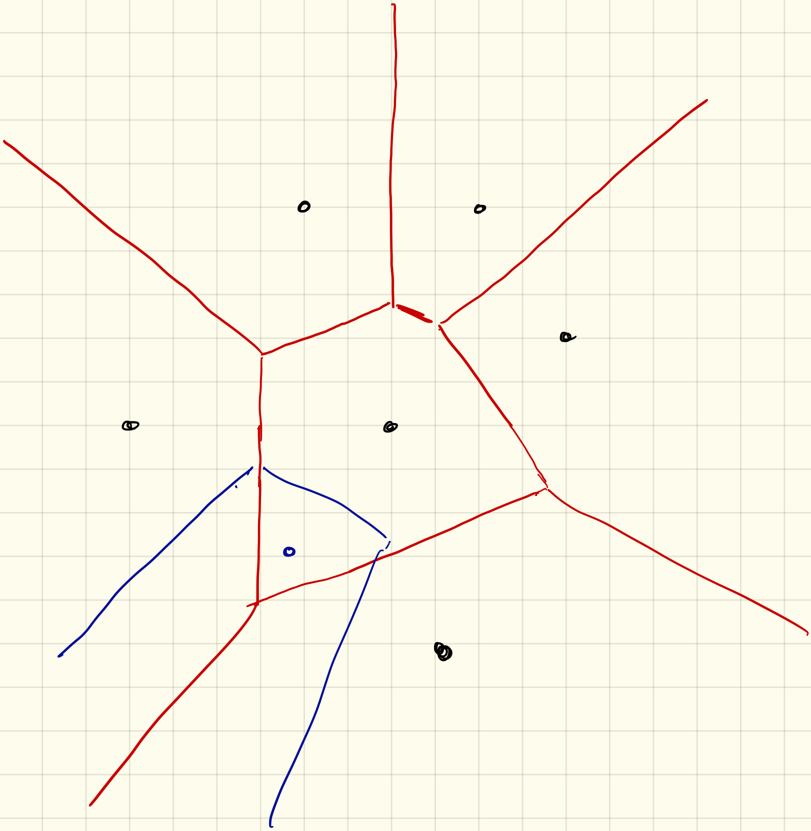
$$Vor(p, P) = \{x : |px| \leq |qx| \ \forall q \in P\}$$

(Note: one cell per point)

Heavily studied:

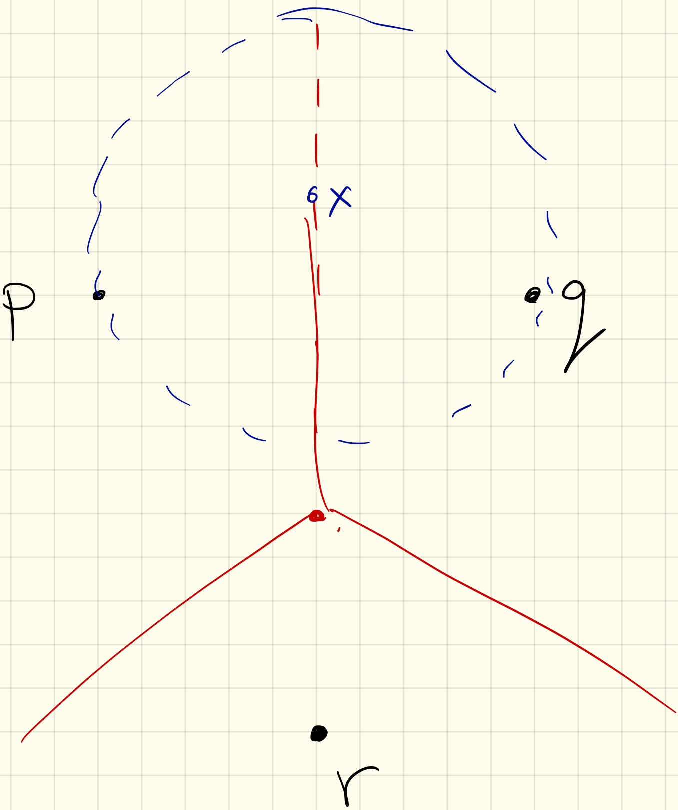
- Many algorithmic approaches.

Best known: incremental



Property

Thm: Consider a point set P and its Voronoi diagram Vor(P).
An edge e is a Voronoi edge $\iff$ for each $\forall x \in e$, the circle centered at x going through the two adjacent sites contains no other sites.

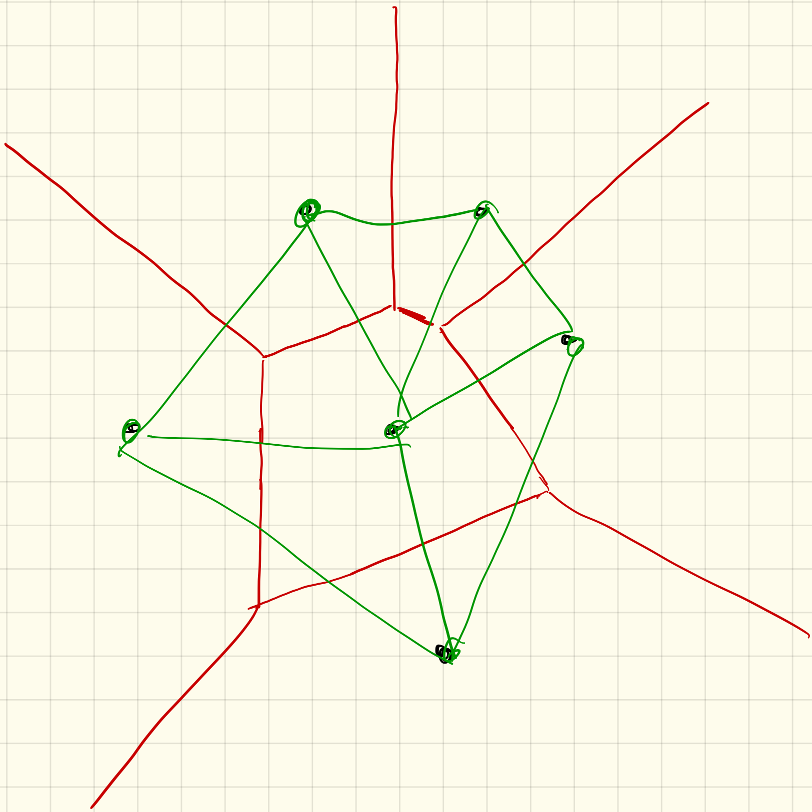


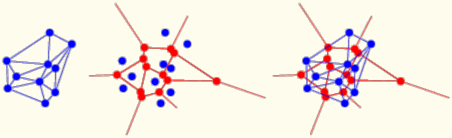
Duality:

Consider the plane.

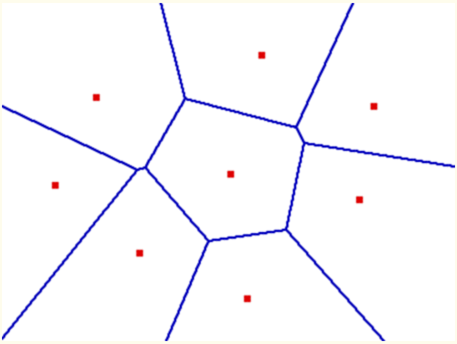
Can make a dual graph:

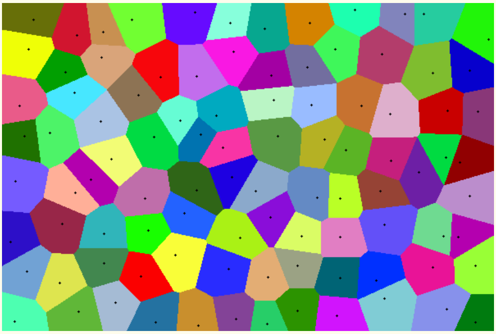
- each face becomes a vertex
- adjacent faces in primal are connected in the dual





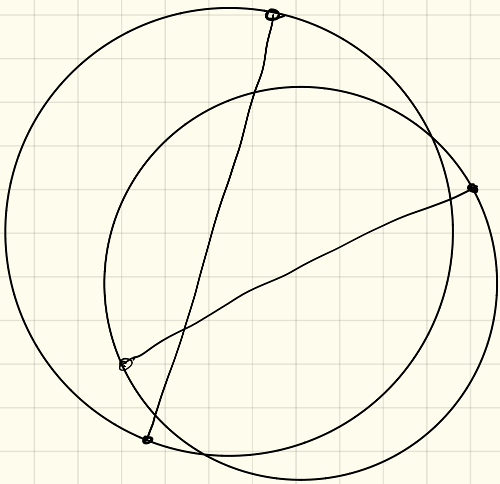
Delaunay triangulation *Voronoi diagram* *Delaunay and Voronoi*





Thm (Delaunay):

Let A & B be 2 circles with chords that properly cross. Then at least one endpoint of one circle's chord is strictly inside other circle.



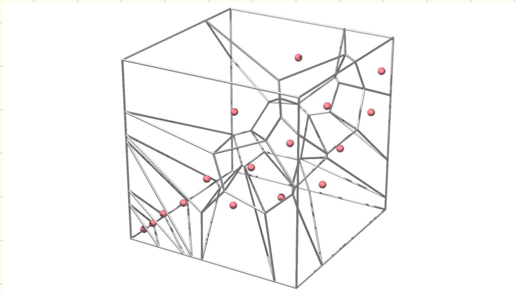
Cor: The straight line dual graph of the Voronoi diagram is the Delauncy triangulation.

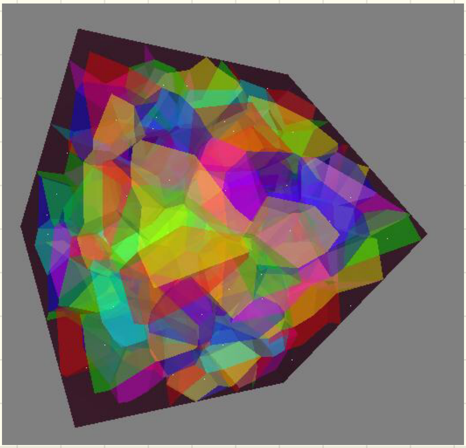
Higher dimensions:

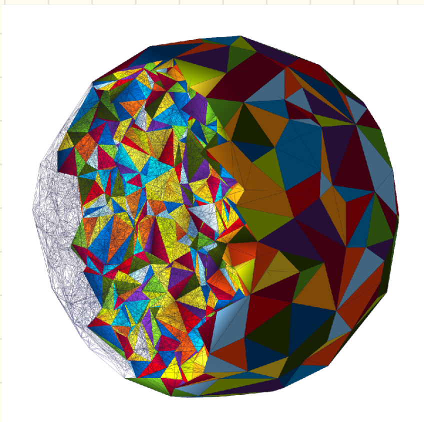
All of these notions generalize:

- The k-dimensional Voronoi diagram is still dual to a k-dimentional Delaunay simplicial complex

This triangulation will be key in surface reconstruction.





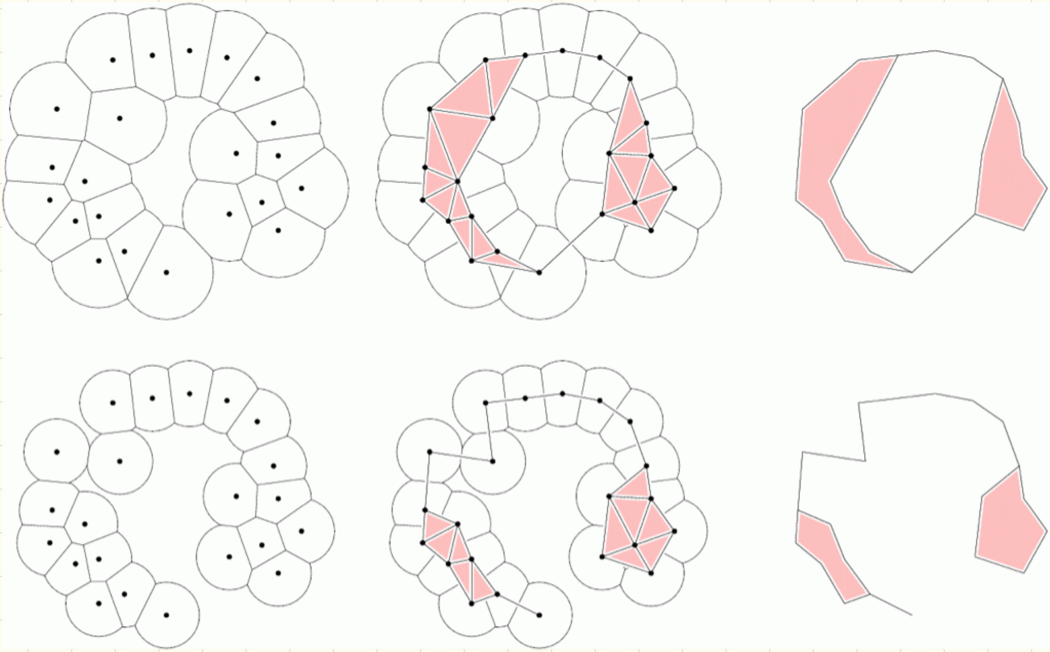


Alpha shapes

Another useful object is the α-shape, which is a subcomplex of $Del(P)$.

Idea: Put balls of radius α around each site.

The α-hull is the union of these disks.



Definition: $C_\alpha(P)$, the α-complex:

$C_\alpha(P)$ is a complex where a simplex $\Delta_T \in Del(P)$ is in $C_\alpha(P)$ if:

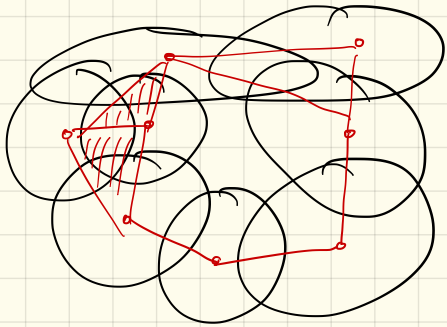
- Δ_T's circumsphere is empty and has radius smaller than α
- or Δ_T is a face of some other simplex in $C_\alpha(S)$

Nerve of a covering:

Let X be a topological space & $\mathcal{U} = \{U_i\}_{i \in I}$ a cover of X.

The *nerve*, $\mathcal{N}\mathcal{U}$, is the abstract simplicial complex with vertex set I where a k-simplex $\{i_0, \dots, i_k\}$ is included $\iff U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_k} \neq \emptyset$.

Picture:



Nerve Theorem

Suppose X + $\mathcal{U}$ are as above, and $\mathcal{U}$ is finite.

Suppose further that nonemty intersections of sets are contractable.

Then $N\mathcal{U}$ is homotopy equivalent to X.

Main use:

Now we can consider these simplicial sets!

Back to α-shapes:

The α-complex has the same homotopy type as the union of balls.

Other complexes:

Čech complex, $\check{C}(X,r)$:

Consider the covering

$$\mathbb{B}_r(X) = \{B_r(x)\}_{x \in X}$$

The complex $\check{C}(X,r)$ is the nerve of this covering:

a simplex of dimension k is included

$\iff$ all k r-balls intersect.

<u>Note</u>: - Nerve theorem applies.

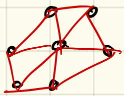
- Nice inclusions:

$$\check{C}(X,r) \subseteq \check{C}(X,r')$$

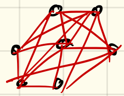
if $r < r'$.

Čech versus α-complex:

The difference is dimension!



α-shape



The Čech complex may contain high dimensional simplices.

The α-complex will <u>not</u>.

Vietoris-Rips Complex

$VR(X, r) =$

$$\{\sigma \subseteq X \mid B_r(x) \cap B_r(y) \neq \emptyset \text{ for all } x, y \in \sigma\}$$

So the Rips complex is any simplex where the diameter is $\leq 2r$.

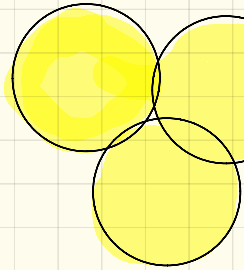
Note:

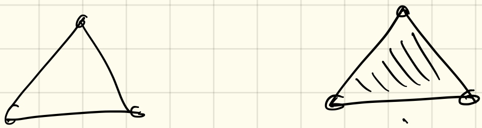
Do not have nerve theorem!

This complex is not a nerve

↳ it is larger.

Difference!





Čech   Rips

<u>But</u>: related.

Always have
$$\check{C}(X,r) \subseteq VR(X,r)$$

In fact, in $\mathbb{R}^n$:

$$\check{C}(X,r) \subseteq VR(X,r) \subseteq \check{C}(X,2r)$$