Generalized Persistent Homology: Part I, Modules

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Goals

- Understand the underlying structure of persistent homology
- Use more general collections of topological spaces, not just filtrations
- Do we have to use homology?

Persistent Homology Recap

Filtration

$$X_0 \to X_1 \to X_2 \to \cdots \to X_n \to \cdots$$

A sequence of topological spaces with maps (often inclusion) between them.

Note: can be indexed by rationals, reals, ...

Homology of Filtration

$$H_k(X_0) \to H_k(X_2) \to \cdots \to H_k(X_n) \to \cdots$$

Persistent Homology

$$H_k^p(X_t) = im(H_k(X_t) \to H_k(X_{t+p})) = im(f_{t,t+p})$$

where $f_{\alpha,\beta} : H_k(X_\alpha) \to H_k(X_\beta)$ is the map induced by the include $X_\alpha \to X_\beta$.

Births and Death Times

Birth

An cycle $c \in H_k(X_t)$ has *birth time* t if $c \notin im(H_k(X_s) \to H_k(X_t))$ any $s < t$.

Death

The *death time* of c is the smallest u such that the map $f_{t,u} : H_k(X_t) \to H_k(X_u)$ maps u to 0.

Persistence Module

Definition

$\mathcal{PH}_k(\mathcal{X})$ is the submodule of $H_k(X_0) \oplus H_k(X_1) \oplus \cdots \oplus H_k(X_n)$ generated by elements of the form $(0, \ldots, 0, c, f_{\alpha,\alpha+1}(c), \ldots, f_{\alpha,\beta}(c) = 0, \ldots 0)$ where $c \in H_k(X_\alpha)$ has birthtime α.

Note: this is equivalent to the original definition (due to Carlsson and Zomordian) of the persistence module as a graded $\mathbb{F}[t]$-module.

Krull-Remak-Schmidt

Theorem

If M is a Noetherian Artinian module the M decomposes uniquely into direction summands

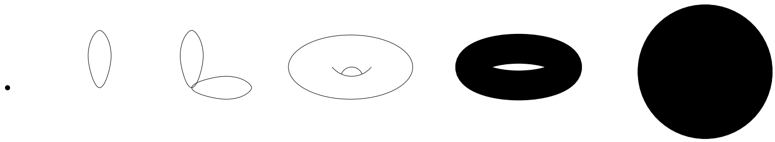
$$M \cong M_1 \oplus \cdots \oplus M_n$$

Recall that the standard persistence algorithm calculates birth and death pairs. Each of these pairs is a summand in the decomposition of the persistence module.

$$\mathcal{PH}_k(\mathcal{X}) = \bigoplus_i \mathbb{F}_{(b_i,d_i)}$$

where $\mathbb{F}_{(b,d)} = 0 \oplus \cdots \oplus \mathbb{F} \oplus \cdots \mathbb{F} \oplus 0 \oplus \cdots \oplus 0$ has non-zero terms for $b \leq t < d$.

Example



$$\begin{aligned}\mathcal{PH}_0(\mathcal{X}) &= \mathbb{F}_{(0,\infty)} \\ \mathcal{PH}_1(\mathcal{X}) &= \mathbb{F}_{(1,4)} \oplus \mathbb{F}_{(2,5)} \\ \mathcal{PH}_2(\mathcal{X}) &= \mathbb{F}_{(3,4)}\end{aligned}$$

Quiver Representation

Quiver

A multi-digraph (Directed graph with multiple edges and loops)

Quiver Representation

Given a quiver $G = (V, E)$, a representation has

- A vector space W_u for each $u \in V$
- A linear map $f : W_u \to W_v$ for each $(u, v) \in E$

Quiver Representation: Examples

Standard Persistence

$H_k(X_0) \to H_k(X_2) \to \cdots \to H_k(X_n) \to \cdots$

The persistence module is a quiver representation.

Zig-Zag Persistence (Carlsson-de Silva)

$H_k(X_0) \leftrightarrow H_k(X_2) \leftrightarrow \cdots \leftrightarrow H_k(X_n) \leftrightarrow \cdots$

Each arrow goes left or right.

Quiver Representation: Examples

Multi-dimensional Persistence (Carlsson-Zomordian)

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
H_k(X_{1,3}) & \longrightarrow & H_k(X_{2,3}) & \longrightarrow & H_k(X_{3,3}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
H_k(X_{1,2}) & \longrightarrow & H_k(X_{2,2}) & \longrightarrow & H_k(X_{3,2}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
H_k(X_{1,1}) & \longrightarrow & H_k(X_{2,1}) & \longrightarrow & H_k(X_{3,1}) & \longrightarrow & \cdots
\end{array}
$$

Quiver Representation: Examples

DAG Persistence (Chambers-L)

$$
\begin{array}{ccccc}
H_k(X_1) & & \longrightarrow & & H_k(X_5) \\
& \searrow & & & \\
& & H_k(X_3) & \longrightarrow & H_k(X_6) \\
& \nearrow & & \searrow \!\!\!\! \nearrow & \\
H_k(X_2) & \longrightarrow & H_k(X_4) & \longrightarrow & H_k(X_k)
\end{array}
$$

Gabriel's Theorem

Theorem

If the underlying undirected graph is an ADE Dynkin diagram that there are finitely many possible irreducible submodules of a quiver representation.

Standard and zig-zag is a Type A Dynkin diagram and irreducible submodules are all of the form $\mathbb{F}_{(b,d)}$.

Decompositions in DAG Persistence