Generalized Persistent Homology: Part II, Why Homology?

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Goals

- Understand the underlying structure of persistent homology
- Use more general collections of topological spaces, not just filtrations
- Do we have to use homology?

Persistent Homology Recap

Filtration

$$X_0 \to X_1 \to X_2 \to \cdots \to X_n \to \cdots$$

A sequence of topological spaces with maps (often inclusion) between them.
Note: can be indexed by rationals, reals, ...

Homology of Filtration

$$H_k(X_0) \to H_k(X_2) \to \cdots \to H_k(X_n) \to \cdots$$

Persistent Homology

$$H_k^p(X_t) = im(H_k(X_t) \to H_k(X_{t+p})) = im(f_{t,t+p})$$

where $f_{\alpha,\beta} : H_k(X_\alpha) \to H_k(X_\beta)$ is the map induced by the include $X_\alpha \to X_\beta$.

Births and Death Times

Birth

An cycle $c \in H_k(X_t)$ has *birth time* t if $c \notin im(H_k(X_s) \to H_k(X_t))$ any $s < t$.

Death

The *death time* of c is the smallest u such that the map $f_{t,u} : H_k(X_t) \to H_k(X_u)$ maps u to 0.

Persistence Module

Definition

$\mathcal{PH}_k(\mathcal{X})$ is the submodule of $H_k(X_0) \oplus H_k(X_1) \oplus \cdots \oplus H_k(X_n)$ generated by elements of the form $(0, \ldots, 0, c, f_{\alpha,\alpha+1}(c), \ldots, f_{\alpha,\beta}(c) = 0, \ldots 0)$ where $c \in H_k(X_\alpha)$ has birthtime α.

Note: this is equivalent to the original definition (due to Carlsson and Zomordian) of the persistence module as a graded $\mathbb{F}[t]$-module.

Krull-Remak-Schmidt

Theorem

If M is a Noetherian Artinian module the M decomposes uniquely into direction summands

$$M \cong M_1 \oplus \cdots \oplus M_n$$

Recall that the standard persistence algorithm calculates birth and death pairs. Each of these pairs is a summand in the decomposition of the persistence module.

$$\mathcal{PH}_k(\mathcal{X}) = \bigoplus_i \mathbb{F}_{(b_i,d_i)}$$

where $\mathbb{F}_{(b,d)} = 0 \oplus \cdots \oplus \mathbb{F} \oplus \cdots \mathbb{F} \oplus 0 \oplus \cdots \oplus 0$ has non-zero terms for $b \leq t < d$.

Category Theory Basics

What is it? (Abstract non-sense?)

Provides a formal framework for mathematical objects, their properties, maps between them, ...

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Examples

- Sets
- Groups, Abelian groups
- Rings, fields, modules, vector spaces
- Topological spaces
- ...

Category Theory Quick Intro

Definition

A *category* $\mathcal{C}$ has the following:

- $Obj(\mathcal{C})$, the objects
- $Hom(\mathcal{C})$, the morphims (or maps) from on object in $Obj(\mathcal{C})$ to another. If $A, B \in Obj(\mathcal{C})$ then $Hom(A, B)$ is the space of all morphisms from $A \to B$.
- A binary operation $\circ$, called composition, of two morphisms. Formally, if $A, B, C \in Obj(\mathcal{C})$ then $\circ : Hom(A, B) \times Hom(B, C) \to Hom(A, C)$ that satisfies:
  - Associativity, for $f \in Hom(A, B), g \in Hom(B, C), h \in Hom(C, D)$, $f \circ (g \circ h) = (f \circ g) \circ h$.
  - Identity, for any object $A \in Obj(\mathcal{C})$, there is a morphism $1_A \in Hom(A, A)$, the identity morphism, such that for any $f \in Hom(A, B)$, $1_A \circ f = f = f \circ 1_B$

Examples of Categories

Category	Objects	Morphisms
Set	sets	set maps
Top	topological spaces	continuous maps
Mfld	smooth manifold	diffeomorphisms
Grp	group	group homomorphism
Ab	Abelian group	group homomorphism
$\mathbf{Vect}_k$	vector space over the field k	linear maps
DAG	directed acyclic graphs	graph homomorphism
Mod	pairs (R, M) where M is module over the ring R	module homomorphism

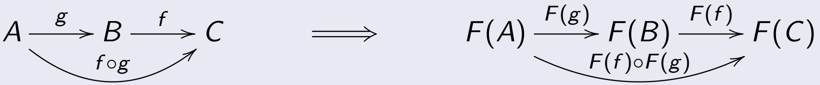
Functors

A map between categories

(Covariant) Functor

$F : \mathcal{C} \to \mathcal{D}$ consisting of

- For any $A \in Obj(\mathcal{C})$, $F(A) \in Obj(\mathcal{D})$
- For any $f \in Hom(A, B)$ where $A, B \in Obj(\mathcal{C})$, $F(f) \in Hom(F(A), F(B))$ such that
  - $F(f \circ g) = F(f) \circ F(g)$
  - $F(1_A) = 1_{F(A)}$



Functors

Arrows swap directions!

Contravariant Functor

$F : \mathcal{C} \to \mathcal{D}$ consisting of

- For any $A \in Obj(\mathcal{C})$, $F(A) \in Obj(\mathcal{D})$
- For any $f \in Hom(A, B)$ where $A, B \in Obj(\mathcal{C})$, $F(f) \in Hom(F(B), F(A))$ such that
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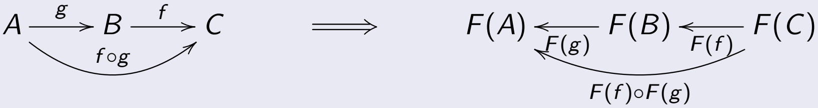

Examples of Functors

- Forgetful functor F : **Grp** $\to$ **Set** that ignores the group structure and restriction on group homomorphism.
- Abelianization function F : **Grp** $\to$ **Ab** that maps $G \to G/[G, G]$.
- Homology operator H_k : **Top** $\to$ **Ab** that sends topological spaces to their k-dimensional homology groups and continuous maps to their maps induced by inclusion.
- Homotopy operator π_k : **Top** $\to$ **Grp** that sends topological spaces to their k-dimensional homotopy groups and continuous maps to their maps induced by inclusion.
- Cohomology operator H^k : **Top** $\to$ **Ab** that sends topological spaces to their k-dimensional homology groups and continuous maps to their maps induced by inclusion. This is a contravariant functor.

Filtrations as Functors

A (graph) filtration can be thought of as a functor **DAG** $\rightarrow$ **Top**.

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot$$

$$\Downarrow$$

$$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \rightarrow X_5 \rightarrow X_6$$

and the corresponding filtration of homology groups can also be thought of as a functor **DAG** $\rightarrow$ **Ab**.

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot$$

$$\Downarrow$$

$$H_k(X_0) \rightarrow H_k(X_1) \rightarrow H_k(X_2) \rightarrow H_k(X_3) \rightarrow H_k(X_4) \rightarrow H_k(X_5) \rightarrow H_k(X_6)$$

Natural Transformations

Definition

If $\mathcal{C}$ and $\mathcal{D}$ are categories with functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\mu : F \to G$ satisfies

- For each $A \in \mathcal{C}$, there is a map $\mu_A : F(A) \to G(A)$
- For each morphism $f \in Hom_{\mathcal{C}}(A, B)$, $\mu_B \circ F(f) = G(f) \circ \mu_A$

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \Big\downarrow{\scriptstyle \mu_A} & & \Big\downarrow{\scriptstyle \mu_B} \\ G(A) & \xrightarrow{G(g)} & G(B) \end{array}$$

Category of Functors

G

iven two categories $\mathcal{C}$ and $\mathcal{D}$ (Require objects of $\mathcal{C}$ to be sets) there is a category $[\mathcal{C}, \mathcal{D}]$ where

- Objects are covariant functors $F : \mathcal{C} \to \mathcal{D}$
- Morphisms are natural transformations $\mu : F \to G$

For example, $[\mathbf{DAG}, \mathbf{Top}]$ and $[\mathbf{DAG}, \mathbf{Ab}]$.

Persistent Homology as a Functor!

$PH_k : [\mathbf{DAG}, \mathbf{Top}] \to [\mathbf{DAG}, \mathbf{Ab}]$ is the functor that maps a filtration of spaces to the corresponding filtration of homology groups.

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot$$

$$\Downarrow$$

$$X_0 \to X_1 \to X_2 \to X_3 \to X_4 \to X_5 \to X_6$$

Apply the persistent homology functor and get:

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot$$

$$\Downarrow$$

$$H_k(X_0) \to H_k(X_1) \to H_k(X_2) \to H_k(X_3) \to H_k(X_4) \to H_k(X_5) \to H_k(X_6)$$

Why Homology?

If F : **Top** $\to \mathcal{C}$ is any functor, this induces a functor

$$PF : [\mathbf{DAG}, \mathbf{Top}] \to [\mathbf{DAG}, \mathcal{C}]$$

- For example, $F = \pi_1$ induces persistent fundamental groups (or $F = \pi_k$ induces persistent homotopy).

$$P\pi_k : [\mathbf{DAG}, \mathbf{Top}] \to [\mathbf{DAG}, \mathbf{Grp}]$$

- PH^k is persistent cohomology
- (Persistent) Alexander module

Note the filtration doesn't need to be an interval graph, it can be any DAG.

What Properties of Persistent Homology Remain?

- Persistent homology groups?
- Barcodes? Persistence diagrams?
- Peristence modules?
- Stability?

For simplicity, we will assume the DAGs are all interval graphs

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot$$

Analogy of Persistent Homology Groups

$$H_k^p(X_i) = im(H_k(X_i)) \to H_k(X_{i+p}))$$

If the functor F : **Top** $\to \mathcal{C}$ is a topological invariant, that is homeomorphic spaces have the same image, then

$$F^p(X_i) = im(F(X_i)) \to im(F(X_{i+p}))$$

For non-interval graphs, can define this in terms of limits and co-limits.

Note: We need to assume $\mathcal{C}$ is a small category.

Persistence Diagrams

Birth time

There is a birth event at time α if $F(X_{\alpha-1} \to X_\alpha)$ is not a epimorphism.

Death time

If a birth event occurs at time α, the corresponding death event occurs at the smallest β such that $im(F(X_{\alpha-1}) \to F(X_\beta)) = im(F(X_\alpha) \to F(X_\beta))$.

Note: We need to assume $\mathcal{C}$ is a small category and that the original filtration is topologically tame. (Only one topological change at any time.)

Peristence diagram

Consists of birth death pairs (α, β) in the plane.

Analogues of Peristence Modules

Modules have two main properties:

- Can add elements
- Can perform scalar multiplication over a base ring
- There operations have the correct unit, associate, commutative and distribution properties.

Additive category

There is a functor $\bigoplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Note: if $\mathcal{C}$ is additive then so is $[\mathbf{Top}, \mathcal{C}]$.

Do we need scalar multiplication?

Decomposing Persistence "Modules"

Unique decompositions

Work in a Krull-Schmidt category, an additive category such that every object is either

- Indecompossible or can be written as a finite direct sum of indecompossibles
- And decompositions are unique: if $X_1 \oplus \cdots \oplus X_m \cong Y_1 \oplus \cdots \oplus Y_n$ then $m = n$ and there exists a permutation π such that $X_{\pi(i)} \cong Y_i$.

When do these decompositions correspond to persistence diagrams?