CS 3100

More LP

Welcome back!

Today:

- HW: may submit by Friday morning
- Next HW - due next Friday
- Sample final coming next week
- Oral grading next Friday (optional)
- Review session: last day of class
- Final: Friday at 8am
  4 cheat sheets

Linear program

In a linear program, we are given a set of variables

The goal is to give these real values so that:

1. We satisfy some set of linear equations or inequalities

2. We maximize or minimize some linear objective function

An example: Maximize profit

A chocolate shop produces 2 products

- Type 1, worth \$1 each
- Type 2, worth \$6 each

Constraints:

- Can only produce 200 of type 1 per day
- And at most 300 of type 2
- Total output per day of both is ≤ 400

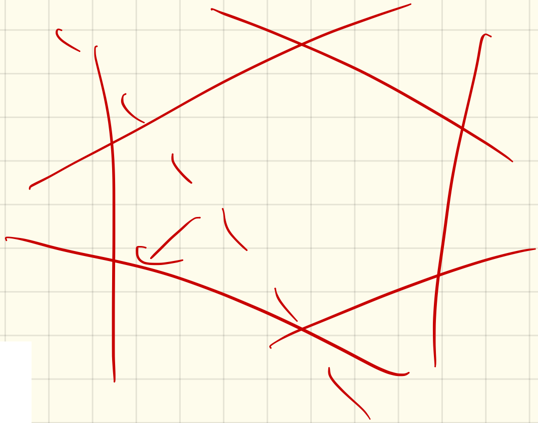
LP:

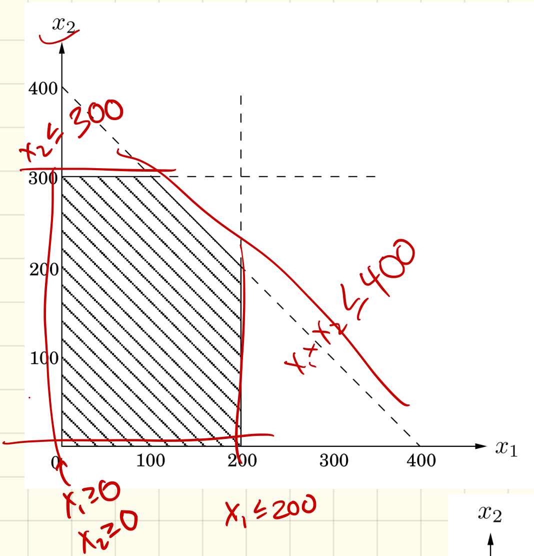
maximize: $x_1 + 6x_2$ ← obj fcn

s.t.

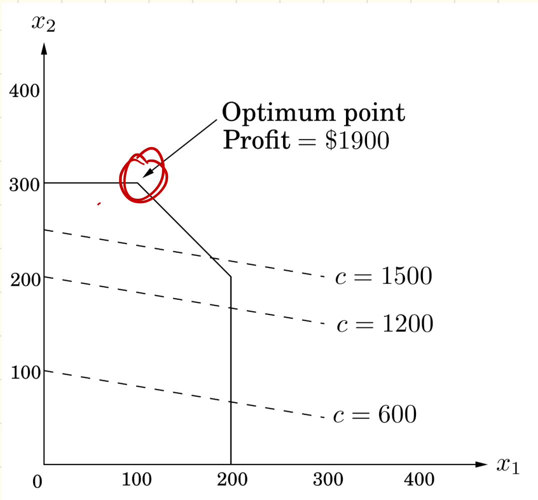
$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$

LP:





x_2
400
$x_2 \le 300$
300
200
$x_1 + x_2 \le 400$
100
0
100
200
300
400
x_1
$x_1 \ge 0$
$x_2 \ge 0$
$x_1 \le 200$

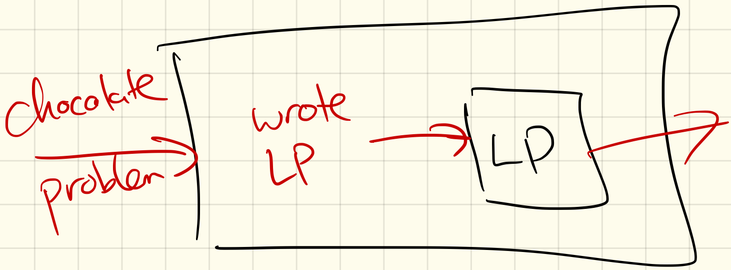


x_2
400
Optimum point
Profit = \$1900
300
200
$c = 1500$
$c = 1200$
100
$c = 600$
0
100
200
300
400
x_1

Connections to other problems:

It turns out that LPs are powerful enough to express <u>many</u> types of problems.

In a sense, we solve many problems by <u>reducing</u> them to an LP:



Ex: Flows & Cuts

Input: directed G w/ edge capacities $c(e)$ $c(u \to v)$

& $s, t \in V$

Goal: Compute flow $f: E \to \mathbb{R}$

s.t.

① $0 \le f(e) \le c(e)$

② $\forall v \ne s, t,$

$$\underbrace{\sum_u f(u \to v)}_{\text{flow into } v} = \underbrace{\sum_w f(v \to w)}_{\text{flow out of } v}$$

Make an LP:

"variable": flow $f(e)$ on each edge

Maximize:

$$\sum_u f(s \to u) \leftarrow [\text{flow out of } s]$$

s.t. $\forall u \to v,$ $f(u \to v) \ge 0$

$f(u \to v) \le c(u \to v)$

Related: Min cuts (S,T)

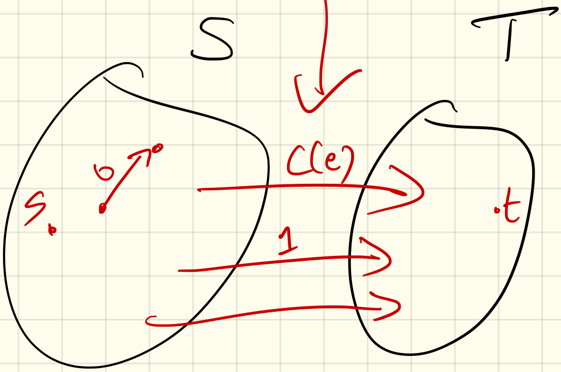
Use indicator variables:

$S_v = 0$ or 1

 if $v \in T$ $\quad$ if $v \in S$

$X_{u \to v} = 1$ if $u \in S$ and $v \in T$

G



The LP:

Minimize $\sum_{u \to v} c_{u \to u} \cdot x_{u \to v}$

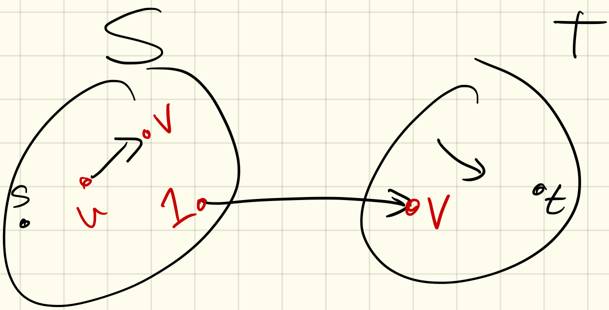
s.t. for each $u \to v$ cut

$$x_{u \to v} + s_v - s_u \geq 0 \quad \forall u \to v$$

$$x_{u \to v} \geq 0 \quad \forall u, v$$

$$s_s = 1$$

$$s_t = 0$$



Note:

For that example, a solution to flow/cuts would yield optimal LP solution.

The reverse is not obvious!

LP might have strange fractional answer which doesn't describe a cut.

It can be shown that this won't happen

↳ but not obvious...

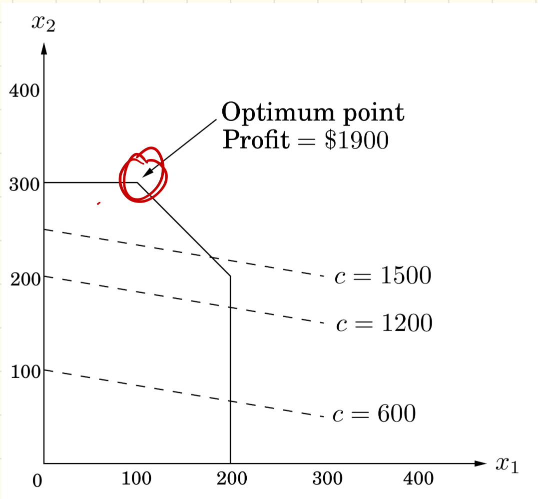
Duality:

Recall our chocolate:

LP: $\max \; x_1 + 6x_2$

s.t.

$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$



Can we check that this is best?

$$\max \quad x_1 + 6x_2$$

s.t.

$$x_1 \le 200 \quad (1)$$

$$\rightarrow x_2 \le 300 \quad (2)$$

$$x_1 + x_2 \le 400 \quad (3)$$

$$x_1, x_2 \ge 0 \quad (4\text{-}5)$$

Play w/ inequalities:

$$(1) + \underbrace{6 \cdot (2)}:$$

$$x_1 \le 200$$

$$6x_2 \le 1800$$

$$\hookrightarrow \le 200 + 1800 = 2000$$

Interesting!

These 2 inequalities tell us that we couldn't ever beat \$2000.

But recall soln was \$1900 — can we get a better combo?

$$\max \quad x_1 + 6x_2$$

s.t.

$$x_1 \le 200 \quad (1)$$

$$x_2 \le 300 \quad (2)$$

$$x_1 + x_2 \le 400 \quad (3)$$

$$x_1, x_2 \ge 0$$

find the magic multipliers

Play: $0 \cdot (1) + 5 (2) + 1 (3)$

$$5(x_2 \le 300)$$

$$\boxed{5x_2 \le 1500}$$

$$\boxed{1(x_1 + x_2 \le 400)}$$

add $x_1 + 6x_2 \le 1900$

These multipliers, $(0, 5, 1)$, are a certificate of optimality.

↳ No valid solution can ever beat \$1900

But how do we find these magic values??

In this, we had three "$\leq$" inequalities

↳ so goal is to find the right 3 multipliers:

$$y_1 \,,\; y_2 \,, \text{and } y_3$$

Multiplier		Inequality
y_1	$\times$	$x_1 \leq 200$
y_2	$\times$	$x_2 \leq 300$
y_3	$\times$	$x_1 + x_2 \leq 400$

Result: $y_1 (x_1 \leq 200)$
$y_2 (x_2 \leq 300)$
$y_3 (x_1 + x_2 \leq 400)$

$$(y_1 + y_3)\, x_1 + (y_2 + y_3)\, x_2 \leq 200 y_1 + 300 y_2 + 400 y_3$$

Note: Make left side look like the original max/min goal, so right will be an upper bound

So here:

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

Means:

$$x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3$$

if: $\begin{cases} y_1, y_2, y_3 \geq 0 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \end{cases}$

Any y_i's would give an upper bound!

We want the best one

↳ ie minimize another LP!

Duality:

$$\max \; 1 \cdot x_1 + 6x_2$$
s.t.
$$x_1 \le 200$$
$$x_2 \le 300$$
$$x_1 + x_2 \le 400$$
$$x_1, x_2 \ge 0$$

$\Updownarrow$ Dual

$$\min \; 200y_1 + 300y_2 + 400y_3$$
s.t.
$$y_1 + y_3 \ge 1$$
$$y_2 + y_3 \ge 6$$
$$y_1, y_2, y_3 \ge 0$$

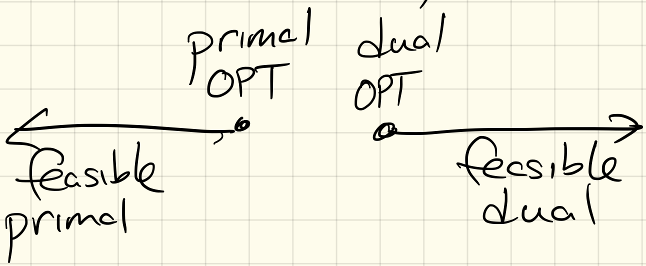
Any solution to bottom is upper bnd to top LP.

$\Rightarrow$ If we can find primal/duals that are equal, both are OPT

Here, 1900: primal $(x_1, x_2) = (100, 300)$
Dual: $(y_1, y_2, y_3) = (0, 5, 1)$

This is just like max flow/min cut duality, in a way.

Works for *any* LP!



this gap — the *duality gap* — is $= 0$.

In general:

Primal LP	Dual LP
$\max\ \vec{c}^T\vec{x}$	$\min\ y^T b$
s.t.	s.t.
$A\vec{x} \le \vec{b}$	$y^T A \ge c^T$
$\vec{x} \ge \vec{0}$	$y \ge 0$

Recall our chocolate:

$\max\ x_1 + 6x_2$	$\min\ 200y_1 + 300y_2 + 400y_3$
s.t.	s.t.
$x_1 \le 200$	$y_1 + y_2 \ge 1$
$x_2 \le 300$	$y_2 + y_3 \ge 6$
$x_1 + x_2 \le 400$	$y_1, y_2, y_3 \ge 0$
$x_1, x_2 \ge 0$	