# CSCI 3100

## Linear Programming

Today:

- No class in 1 week
  → reading assignment
- HW due by end of day on Wednesday

# Linear program

In a linear program, we are given a set of variables

The goal is to give these real values so that:

1. We satisfy some set of linear equations or inequalities

2. We maximize or minimize some linear objective function

## An example: Maximize profit

A chocolate shop produces 2 products

- Type 1, worth \$1 each
- Type 2, worth \$6 each

**Constraints:**

- Can only produce 200 of type 1 per day
- And at most 300 of type 2
- Total output per day of both is $\leq 400$

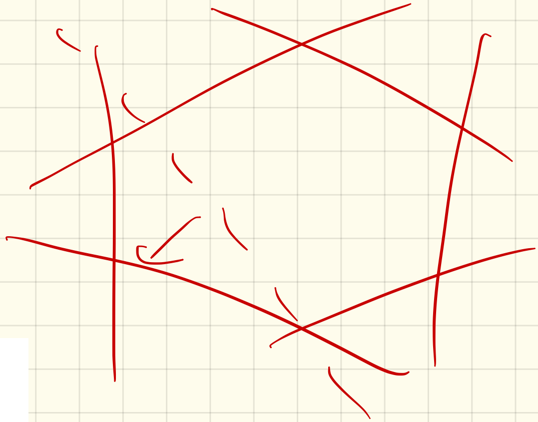
**LP:**

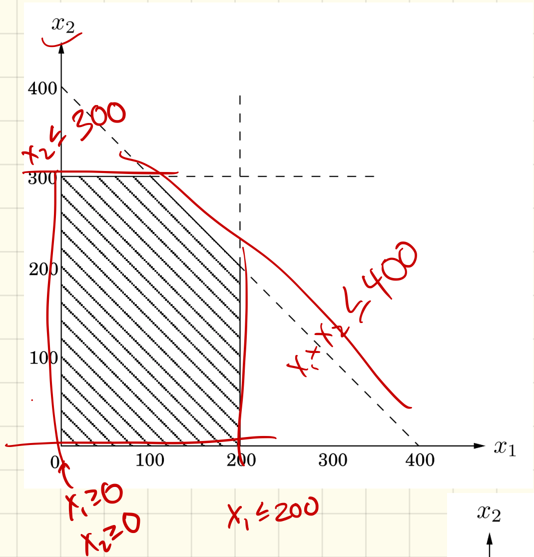
$$\text{maximize: } \underbrace{x_1 + 6x_2}_{\text{obj fcn}}$$

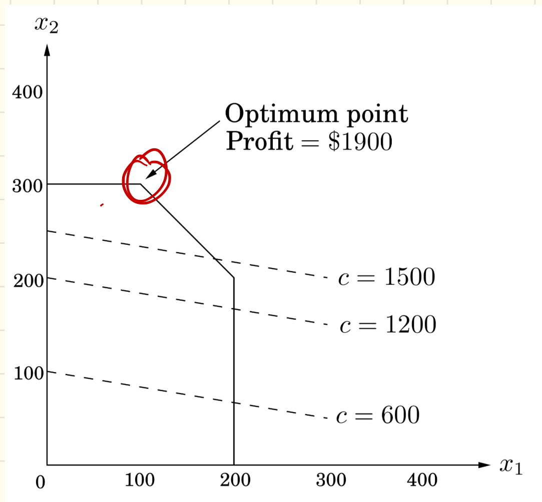
s.t.

$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$

# LP:




$x_2$
400
300
200
100
0
100
200
300
400
$x_1$
$x_2 \le 300$
$x_1 + x_2 \le 400$
$x_1 \ge 0$
$x_2 \ge 0$
$x_1 \le 200$


$x_2$
400
300
200
100
0
100
200
300
400
$x_1$
Optimum point
Profit = $1900
$c = 1500$
$c = 1200$
$c = 600$

These go up in dimention with more $x_i$'s:

new chocolate!

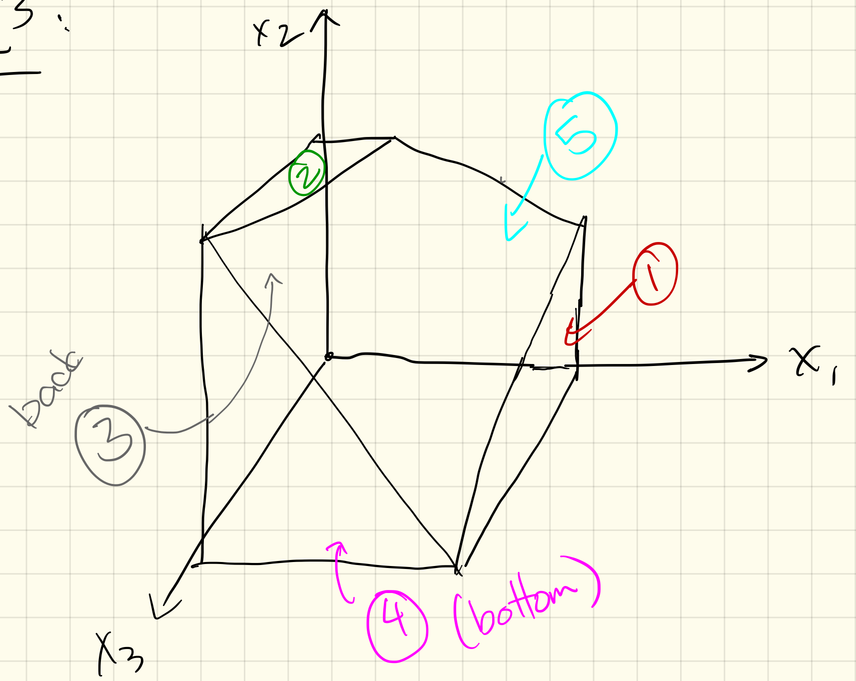
Maximize $x_1 + 6x_2 + 13x_3$
s.t.

$$x_1 \le 200 \quad (1)$$
$$x_2 \le 300 \quad (2)$$
$$x_1 + x_2 + x_3 \le 400$$
$$x_2 + 3x_3 \le 600$$

and

$$x_1 \ge 0 \quad (3)$$
$$x_2 \ge 0 \quad (4)$$
$$x_3 \ge 0 \quad (5)$$

$\mathbb{R}^3$:



Another (more general)

n foods, m nutrients

Let $a_{i,j}$ = amount of nutrient i in food j

$r_i$ = requirement of nutrient i

$x_j$ = amount of food j purchased

$c_j$ = cost of food j

Goal: Buy food so you satisfy nutrients while minimizing cost.

min $c_1 \cdot x_1 + c_2 x_2 \cdots + c_n x_n$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \\ a_{m,1} & \cdots & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

$A\vec{x} \geq \vec{r}$

The LP:

$$\boxed{\begin{aligned} &\min \sum_i x_i \\ &A\,\bar{x} \geq \bar{r} \end{aligned}}$$

$$a_{1,1}\,x_1 + a_{1,2}\,x_2 + \cdots a_{1,n} \cdot x_n \geq r_1$$

$$\vdots$$

In general, get systems like this:

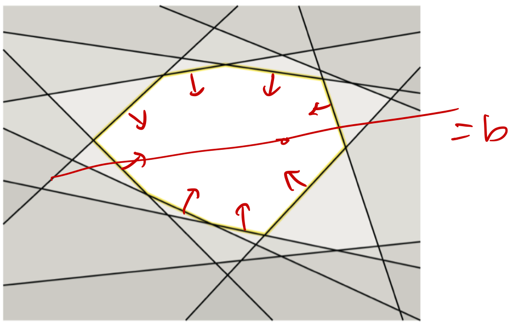
$$\text{maximize} \sum_{j=1}^{d} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{d} a_{ij} x_j \le b_i \quad \text{for each } i = 1 \,..\, p \qquad 1$$

$$\sum_{j=1}^{d} a_{ij} x_j = b_i \quad \text{for each } i = p+1 \,..\, p+q \qquad 2$$

$$\sum_{j=1}^{d} a_{ij} x_j \ge b_i \quad \text{for each } i = p+q+1 \,..\, n \qquad 3$$

Geometric picture:




A two-dimensional polyhedron (white) defined by 10 linear inequalities.

## Canonical Form:

Avoid having both $\leq$ and $\geq$.

So get something more like our first example:

$$\text{maximize} \sum_{j=1}^{d} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for each } i = 1 \,..\, n$$

$$x_j \geq 0 \quad \text{for each } j = 1 \,..\, d$$

Or, given a vector $\vec{c}$ & matrix $A$:

maximize $\vec{c} \cdot \vec{x}$

s.t. $A\vec{x} \leq \vec{b}$

Anything can be put into canonical form:

① Avoid = eqn: $\sum a_i x_i = b$

$\sum a_i x_i \leq b$

$+ \sum a_i x_i \geq b$

② Avoid $\geq$:

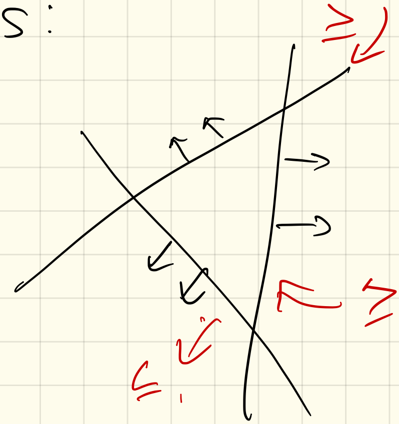
$\left[\sum a_i x_i \geq b\right] \times -1$

$-\sum a_i x_i \leq -b$

$A\bar{x} \leq b$

How could these <u>not</u> have a solution?

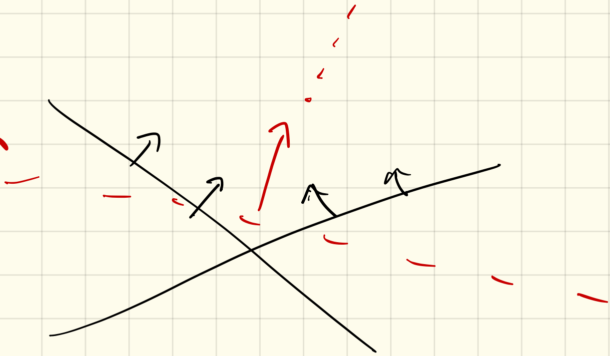
2 ways:




infeasible

or

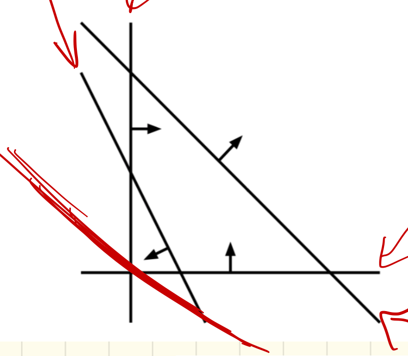
unbounded

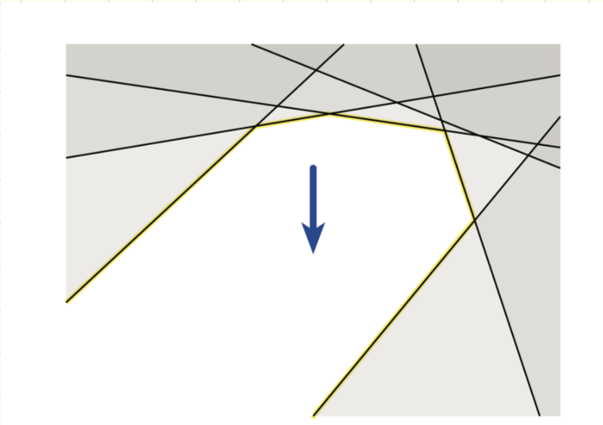


# Better pictures (still 2-d):

$$\begin{aligned} &\text{maximize } x - y \\ &\text{subject to } 2x + y \le 1 \\ &\qquad\qquad\quad x + y \ge 2 \\ &\qquad\qquad\quad x, y \ge 0 \end{aligned}$$

obj





## Note:

1. Multiplying by $-1$ turns any maximization problem into a minimization one:

$$\max \ \vec{c}\,\vec{x} \quad \text{s.t.} \quad A\vec{x} \le \vec{b}$$

$$\min \ -\vec{c}\,\vec{x} \quad \text{s.t.} \quad -A\vec{x} \ge -\vec{b}$$

2. Can turn inequalities into equalities via slack variables:

$$\sum_{i=0}^{n} a_i x_i \le b$$

$$\Downarrow$$

$$a_1x_1 + a_2x_2 + \dots a_nx_n + S = b$$

$$S \ge 0$$

(3) Can change equalities into inequalities, also!

$$\sum_{i=1}^{n} a_i x_i = b$$

$$\Downarrow$$

(already saw)

Solving LP's:
The simplex algorithm

"ball dropping"

Start on boundary (some linear constraint)
if not best possible, improve by moving along this constraint

This will stop at optimal solution