CSCI 3100

Graph NP-Hard problems

Announcements

- No office hours today
- Next HW - more NP-Hardness
  + oral grading

Last time

NP-Hard problems:
- SAT (CIRCUITSAT)
- 3SAT
- Independent Set

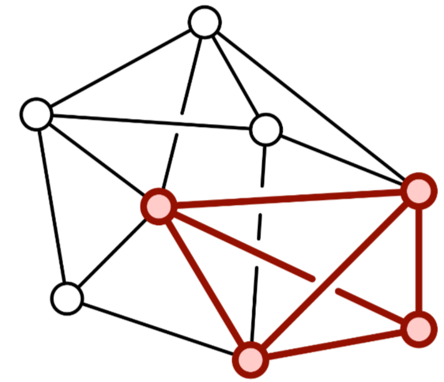
How? Reductions!

To prove any other problem A is NP-Hard, we'll use a reduction:

Reduce a known NP-Hard problem to A.

Next one: Clique

A clique in a graph is a subgraph which is complete - all possible edges are present.



A graph with maximum clique size 4.

How could we check if G has a clique of size k?

Take all size k subgraphs, check if all edge are present b/t those vertices:

$$O(n^k) \rightarrow \binom{n}{k} \cdot k \cdot n = O(kn^{k+1})$$

Decision version: Does G have a clique of size k?

Input: G, k

Output: Yes/No

This is NP-Complete:

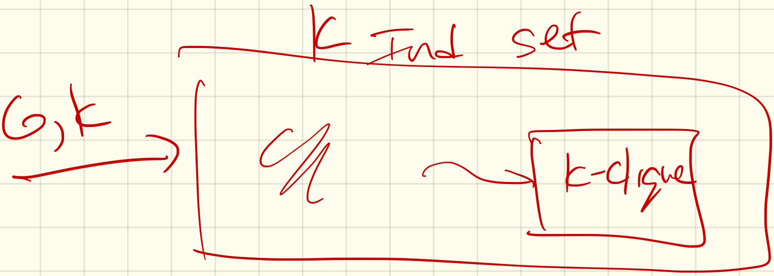
① In NP. Why?

Given the k vertices in the clique, I can verify all edges are present in $O(nk)$.

② NP-Hard:

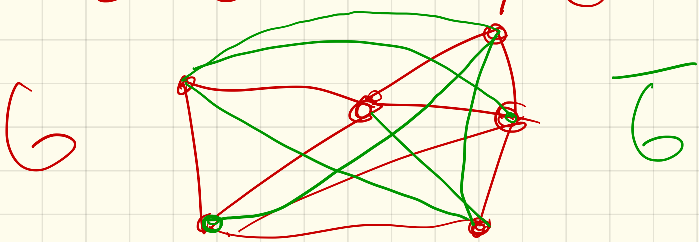
What should we reduce to k-Clique?

Ind. set: Given G & k are there k vertices w/ no edges b/t them?



Given G, create $\overline{G}$, the complement of G:

- $\overline{G}$ will have same vertex set as G
- $e \in \overline{G} \iff e \notin G$



So:

$$
\begin{tikzcd}
\boxed{\text{graph } G} \arrow[r, "\cancel{O(n)}\ O(n^2)"] & \boxed{\text{complement graph } \overline{G}} \arrow[d, Rightarrow, "\textsc{MaxClique}"] \\
\boxed{\text{largest independent set}} & \boxed{\text{largest clique}} \arrow[l, "\text{trivial}"']
\end{tikzcd}
$$

G has ind set of size k $\iff$ $\overline{G}$ has k-clique

pf: if G doesn't have edges b/t k vertices, $\overline{G}$ will (& vice versa)

Conversion: $O(n^2)$ time

Next: Vertex Cover:

A set of vertices which touches every edge in G.

K-Vertex cover (decision version):

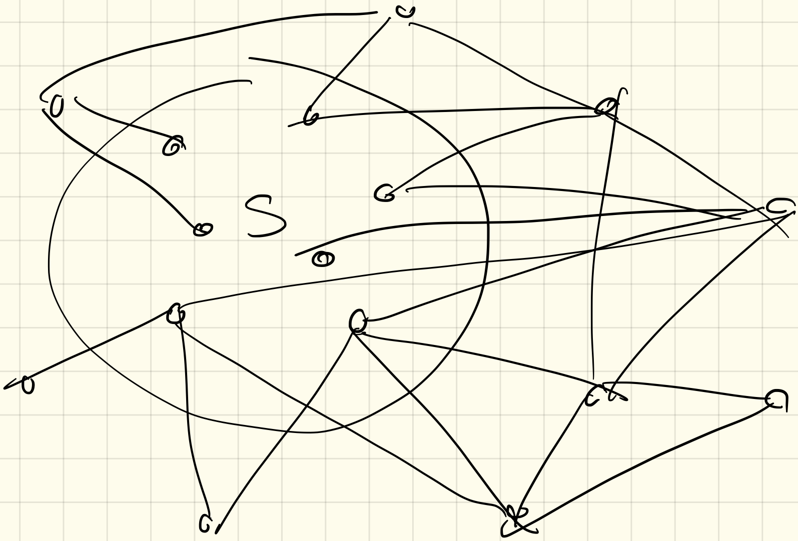
Given G & k, does G have a cover of size k?

In NP:

Given k vertices, check in $O(\overset{O(n^2)}{m})$ time that all edges are "covered."

NP-Hardness: reduce what?
(propably clique or ind set!)

Key: If S is independent set, what is $V-S$?



$V-S$ is a vertex cover!

- edges either go from S to $V-S$, or stay in $V-S$

So simple reduction!

Given G & k to indep. set, ask if $\exists$ vertex cover of size $n-k$.

$\Rightarrow$: Spps ind set of size k $\Rightarrow$ vertex cover of $n-k$

$\Leftarrow$: If cover of size $n-k$, that means no edges b/t any of k vertices not in cover $\Rightarrow$ ind set of size k.

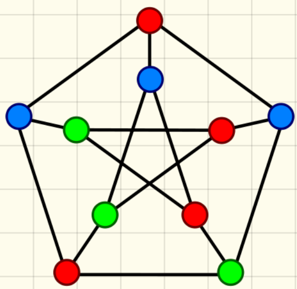
$$
\begin{tikzcd}
\boxed{\text{graph } G=(V,E)} \arrow[r, "\overset{O(1)}{\text{trivial}}"] & \boxed{\text{graph } G=(V,E)} \arrow[d, Rightarrow, "\textsc{MinVertexCover}"] \\
\boxed{\text{largest independent set } V\setminus S} & \boxed{\text{smallest vertex cover } S} \arrow[l, "O(n)"']
\end{tikzcd}
$$

Next: Graph Coloring

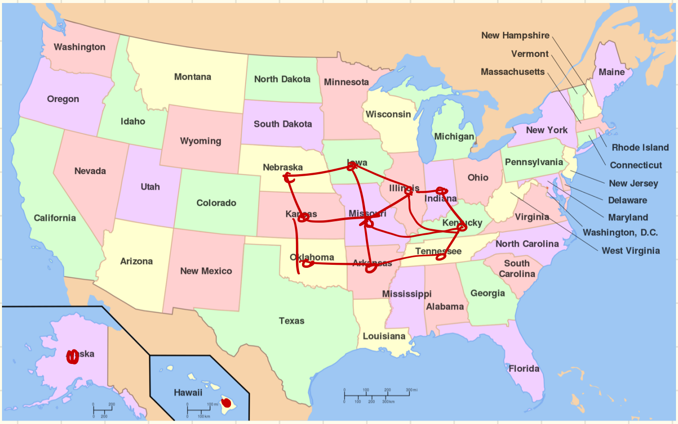
A **k-coloring** of a graph G is a map: $c: V \to \{1, \ldots, k\}$ that assigns one of k "colors" to each vertex so that every edge has 2 different colors at its endpoints.

Goal: Use few colors

Peterson is 3 colorable



Aside: this is famous!
Ever heard of map coloring?



Famous theorem: 4 color thm
Every planar G is 4-colorable.

Thm: 3-colorability is NP-Complete.

(Decision version: Given G, output yes/no)

In NP:

Give you a coloring $c: V \to \{1..3\}$, + in $O(n^2)$ check no edges b/t vertices of same color.

NP-Hard:

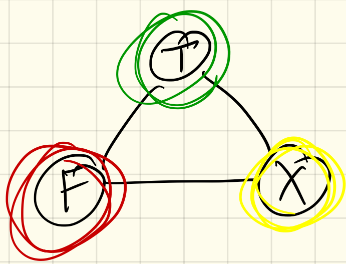
Reduction from 3SAT.

Given formula for 3SAT Φ, we'll make a graph G_Φ.

Φ will be satisfiable $\Longleftrightarrow$ G_Φ can be 3-colored.

Key notion: Build gadgets!

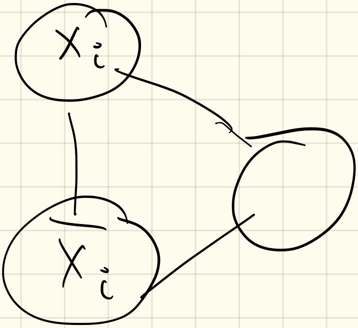
(1) Truth gadget - one



Must use 3 colors - establishes a "true" color.

(2) Variable gadget –

one per SAT variable (n total)

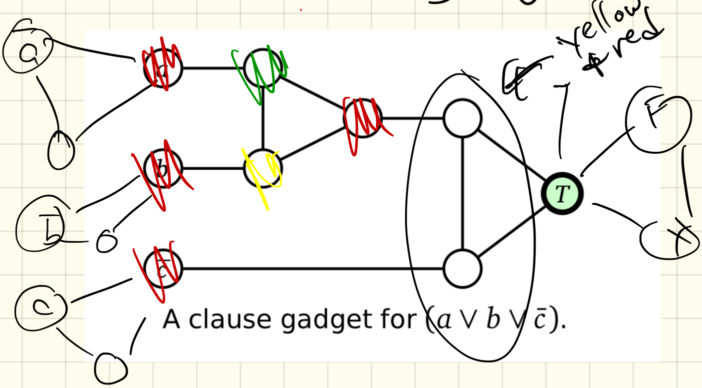


so both x_i + $\overline{x_i}$ can't be true

One of these will be colored true/false & both x_i & $\overline{x_i}$ can't be true

③ Clause gadget:

For each clause, join 3 of the variable vertices to the "true" vertex from the truth gadget.



A clause gadget for $(a \vee b \vee \bar{c})$.

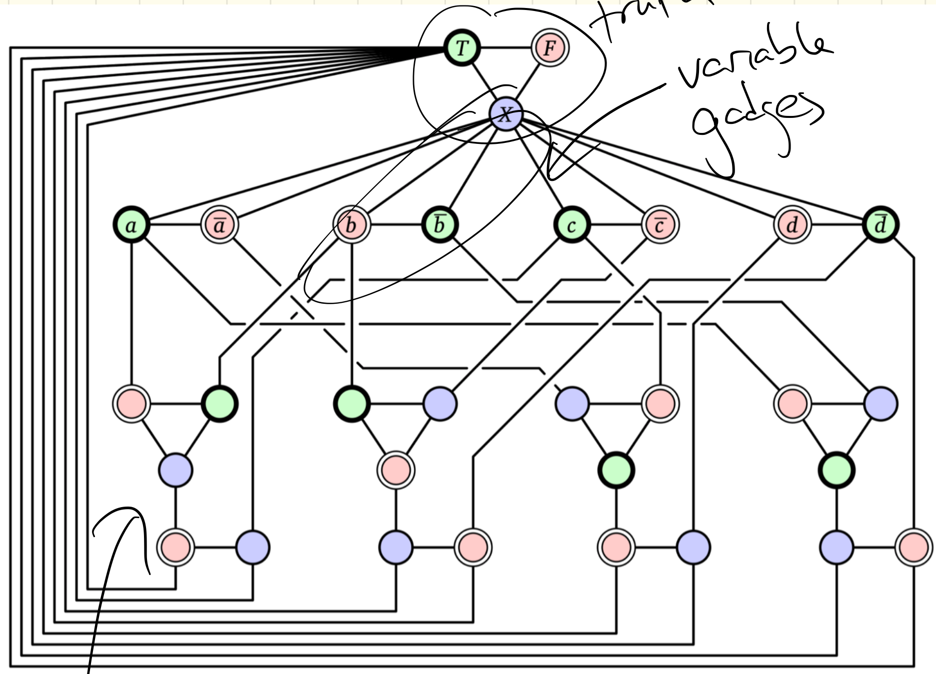
Idea: If all inputs are colored False, can't 3-color:

Case analysis

3 colorings of G_{Φ} $\iff$ Φ is satisfiable

Pf:

Final reduction image:



truth

variable gadges

A 3-colorable graph derived from the satisfiable 3CNF formula
$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$

clause gadges

Time to build G_Φ:

$3n$ on vertex gadgets

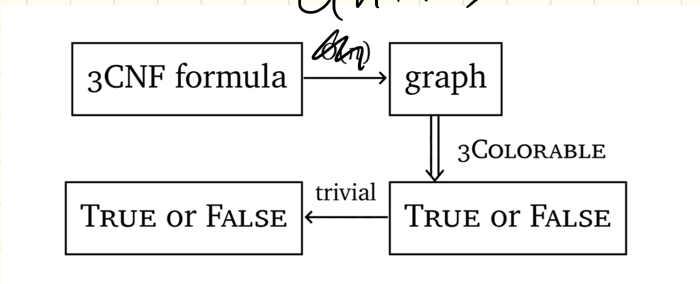
$O(1)$ on truth gadget

$O(m)$ to build clause edges

\# clauses

So:

$O(n+m)$



Next time:

- More reductions
- Plus some non-graph problems!