CS314 — NP-Hardness

Note Title 11/4/2013

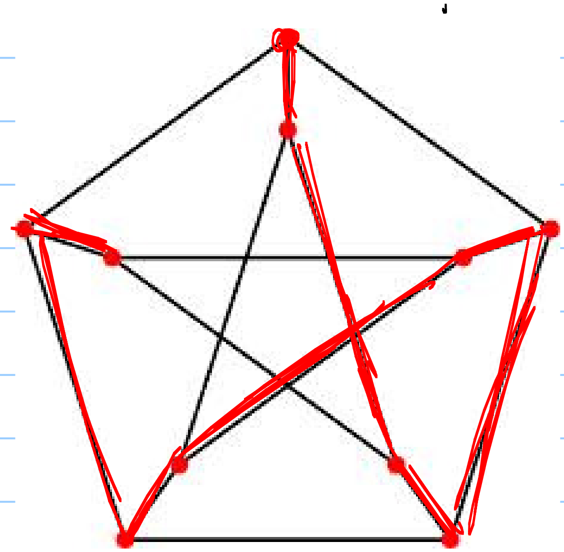
Announcements

- Oral grading next Tuesday
- Survey today

Hamiltonian Cycle

A cycle in a graph which visits each vertex exactly once.

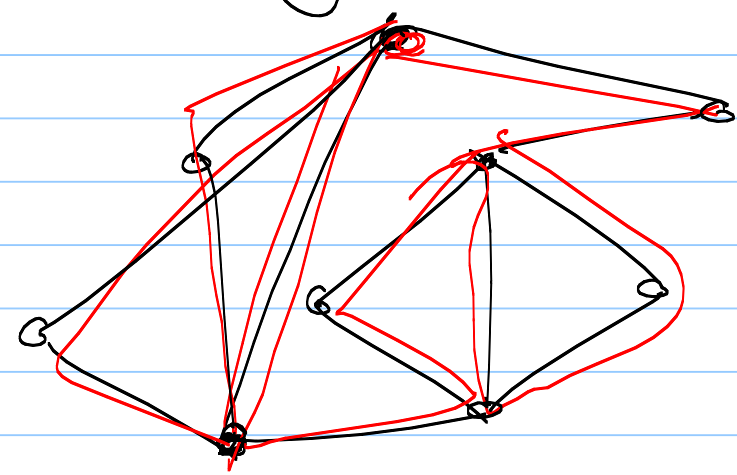
How to find?



Ham. path
(no Ham cycle)
$(n-1)!$

Note: **Not** the same as an Eulerian cycle!

Thm: G has an Euler cycle $\iff$ every vertex of G has even degree.



Q: Does G have a Hamiltonian cycle?

(Yes/no-decision problem)

In NP:

Given an ordering $v_1 \ldots v_n$ check that it is a cycle.

NP-Hard: Reduce vertex cover to Ham. cycle:

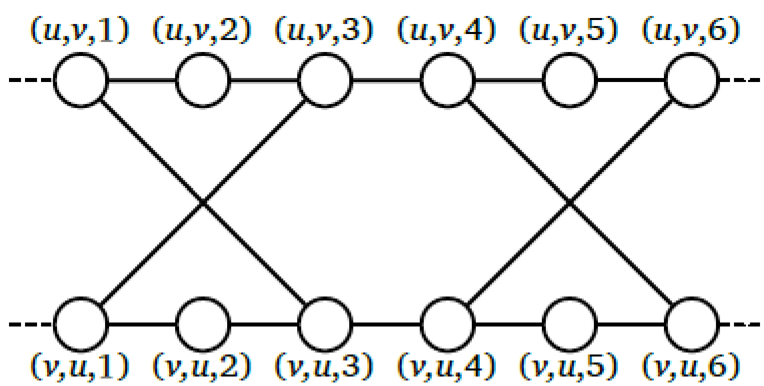
Given a graph G & integer k, answer yes/no if G has a vertex cover of size k.

(Use Ham cycle as a subroutine)

More gadgets!

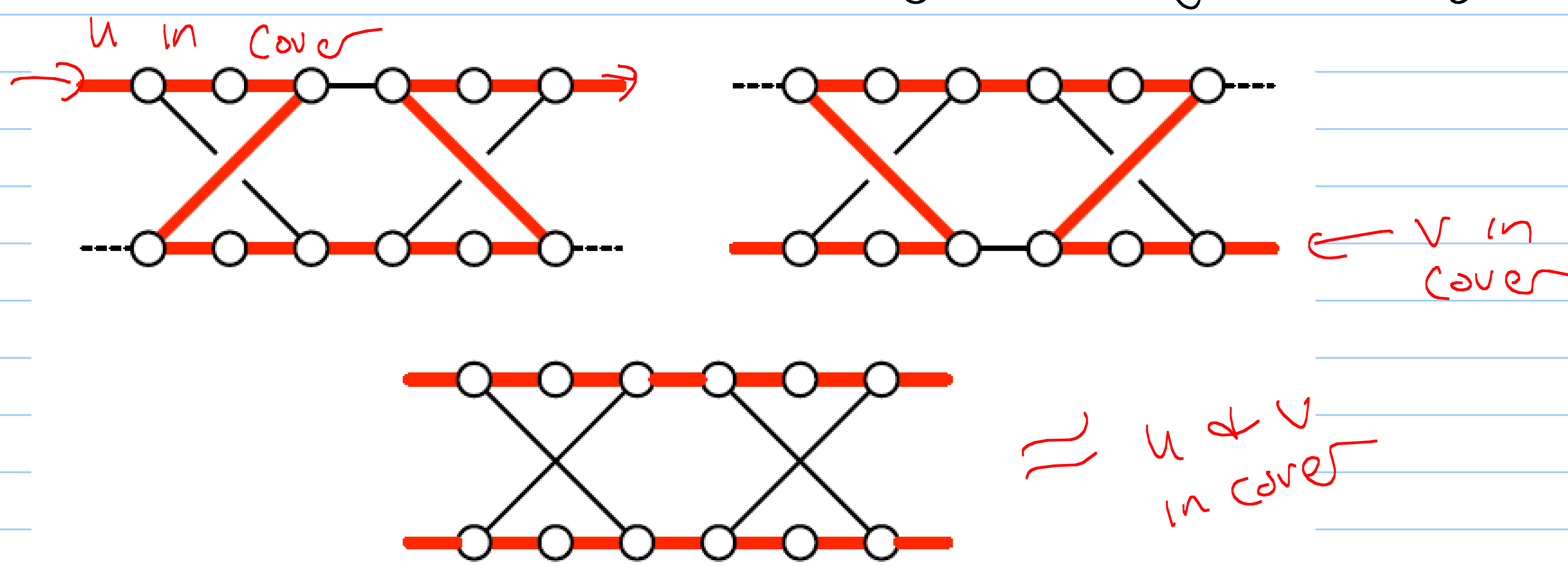
Make a new graph G':

① For each edge uv, edge gadget (EG) in G' with 12 vertices + 14 edges.



The route through this will correspond to u, v, or both u + v being in the cover!

Note: only 3 possible ways for the Ham. cycle to go through:

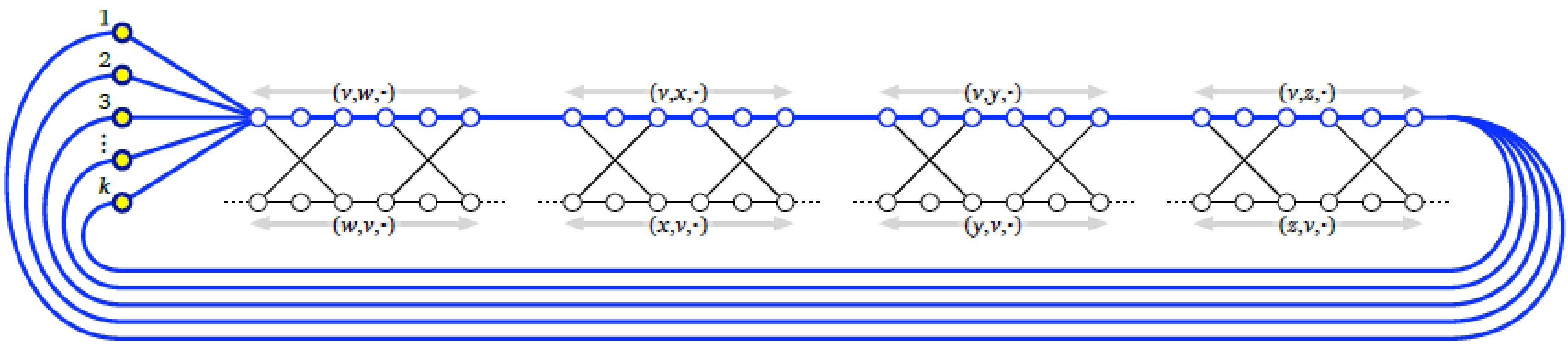


(2) k cover vertices, numbered 1 to k.

(3) For each vertex u $\in G'$, string together all the edge gadgets into a vertex chain.

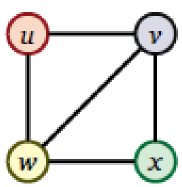
Then connect chains to cover vertices on either end.

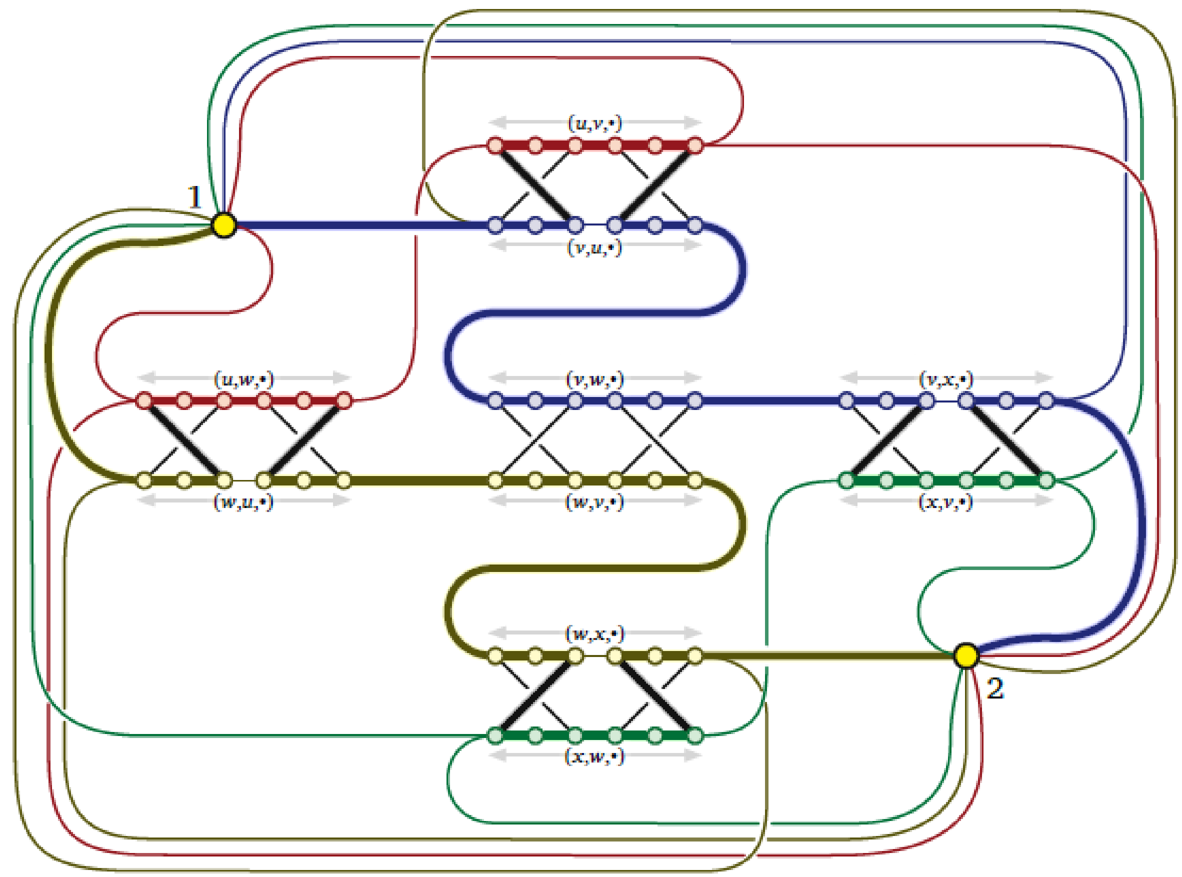
So for a vertex v:



The vertex chain for v: all edge gadgets involving v are strung together and joined to the k cover vertices.

Bigger example:





Now: $\Rightarrow$

If $\{v_1, v_2, \ldots, v_k\}$ is a cover in G, then can get a Ham. cycle in G':

Start at 1, go through vertex chain for v_1

Then go to 2, & chain for v_2 etc.

Harder: $\Leftarrow$

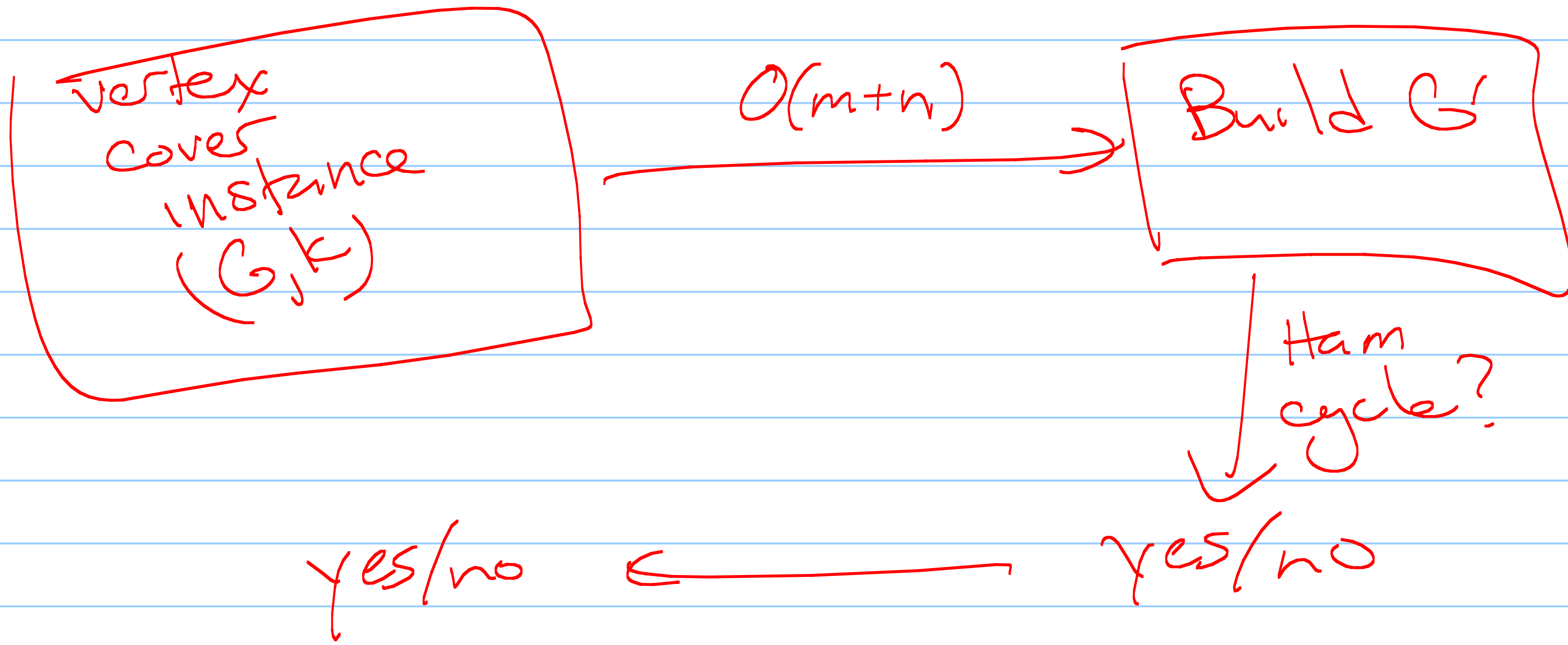
Consider any Ham cycle in G'.

Must alternate cover vertices and vertex chains.

Any vertex chain taken will give the k vertices in a cover.

(A bit more work to do this...)

Ham cycle



Traveling Salesman

Given n cities along with (all) pairwise distances between them, what is the shortest tour of all the cities?

Decision version:

Is there a tour of length $\leq k$?

NP-Hard: Ham cycle to TSP

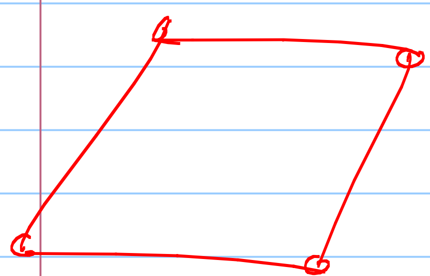
input: unweighted graph G

Construct G': same vertices

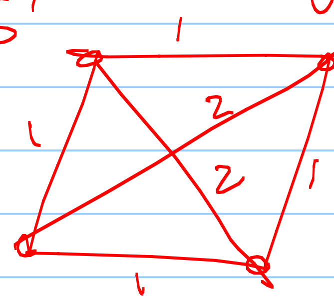
if $e \in G$, put $e \in G'$ with weight 1.

if $e \notin G$, put $e \in G'$ with weight 2.

G



G'



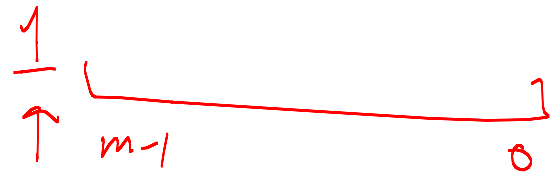
Q: does G' have TSP tour of length n?

Subset Sum

Given a set of numbers $X = \{x_1, x_2, \ldots, x_n\}$, & a target t, is there a subset of X summing to t?

Ex: (see recursion notes!)

$n \cdot t$



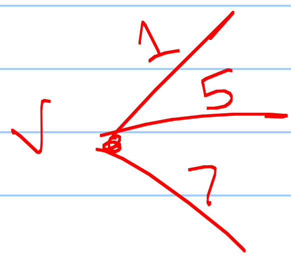
NP-Hard: Reduction from vertex cover:

Given G & k.

- Number G's edges from 0 to $m-1$.
  Put $b_i := 4^i$ in X for each edge i.
- For each vertex v, put

$$a_v := 4^m + \sum_{i \text{ incident to } v} 4^i$$

$$|X| = m+n$$



$$4^m + 4^1 + 4^5 + 4^7$$

$m \quad \underbrace{\qquad\qquad\qquad}_{m-1 \quad\cdots\quad 0}$

So everything in X is a base-4 number:

- m^{th} digit is 1 if it is a vertex
- i^{th} digit is 1 is integer represents edge i or one of its endpoints.

Then set $t = k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i$