CS314 - NP-Hardness

Note Title 11/1/2013

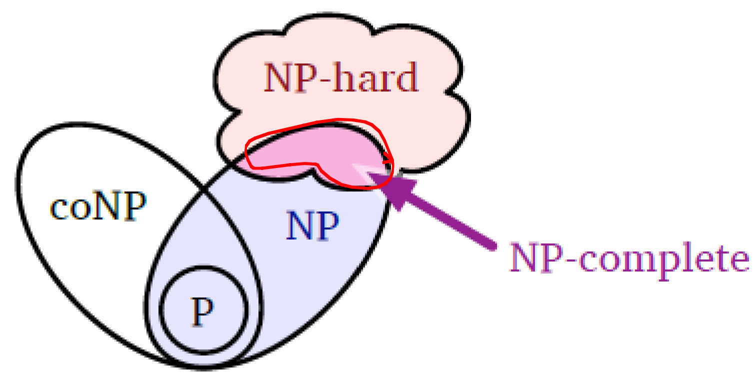
Announcements

- Next ~~hw~~ hw up tonight
  due 1 week from Mon/Tues
  (oral grading)

NP-Completeness

A problem is NP-Complete if it is both:

- in NP
- NP-Hard

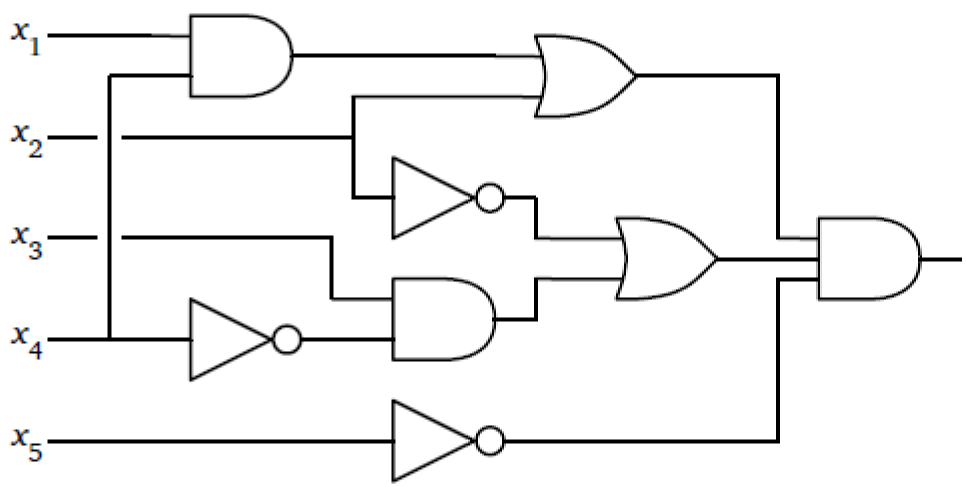


More of what we *think* the world looks like.

polynomal heirarchy

Thm (Cook-Levin) : (The first known NP-Hard problem)

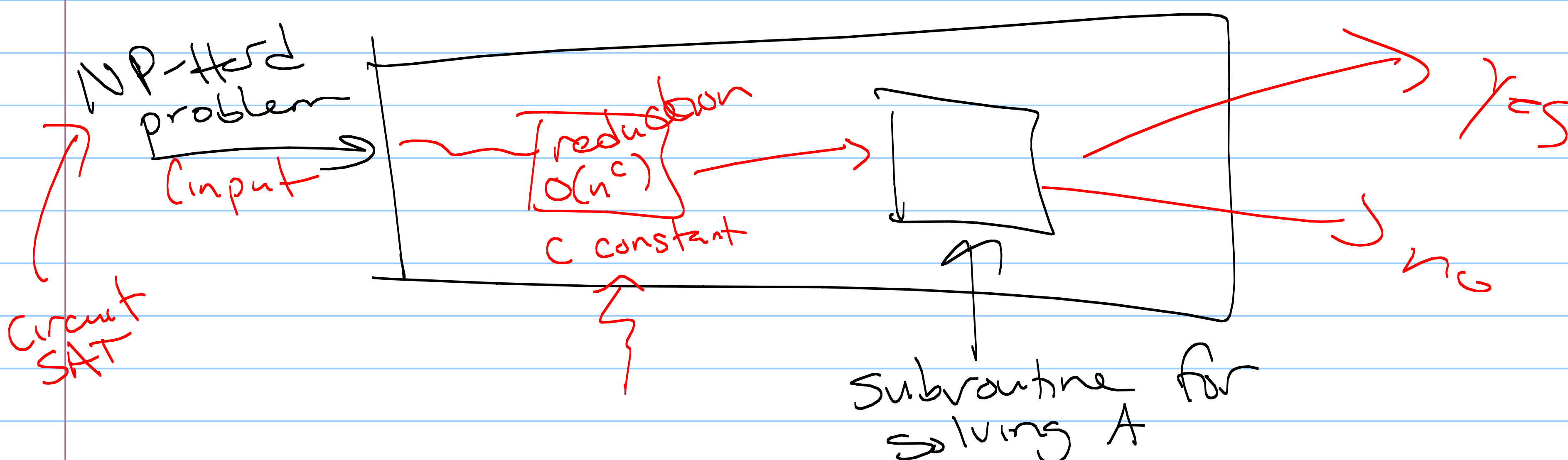
Circuit Satisfiability is NP-Complete.



"pf": any turing machine can be turned into a circuit

To prove NP-Hardness of A:

☆ Reduce a known NP-Hard problem to A.

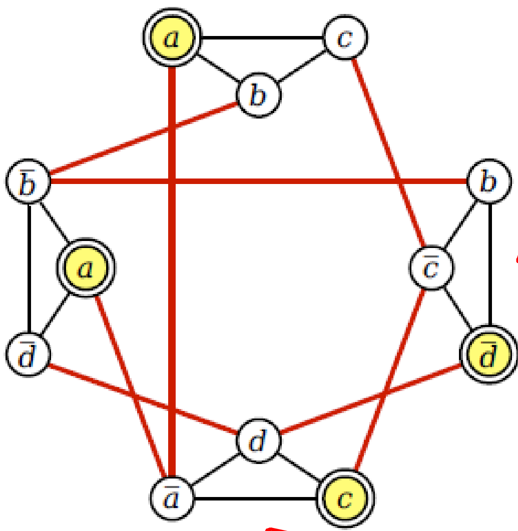


Last time: NP-Hard Problems

- SAT (from circuit SAT)
- 3SAT (from circuit SAT)
- Independent Set (from 3SAT)

How? Transform known hard problem to new one!

Ex: $(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$

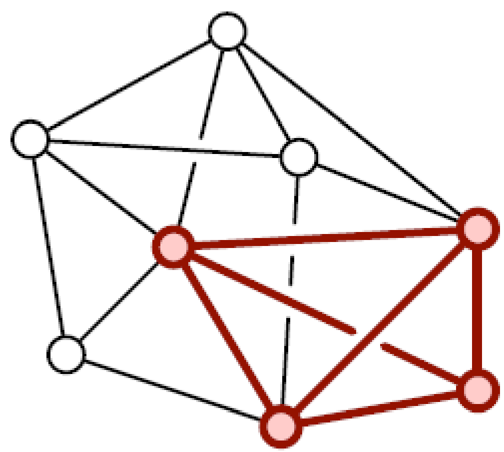


Another: Clique

A clique in a graph is a subgraph which is complete - all possible edges are present.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\approx O(n^k)$$



A graph with maximum clique size 4.

Decision version: Does G have a clique of size k?

NP-Complete:

1. In NP. Why?

How to verify a "yes"?

Given a set k vertices, takes $O(nk)$ time to check if all edges are present.

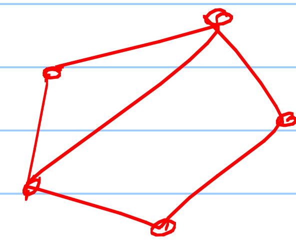
(2) What should we reduce to k-clique?

Ind Set $\rightsquigarrow$ k-clique

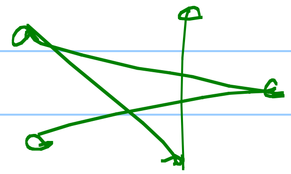
Given G, answer if ind. set of size k is present.

Transform G to $\bar{G}$

G

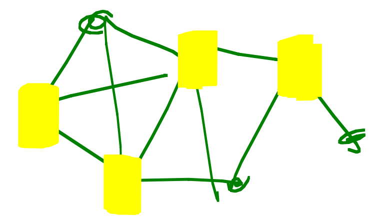


$\bar{G}$



So:

$$\begin{tikzcd}
& \boxed{\text{graph } G} \arrow[r, "O(n)"] & \boxed{\text{complement graph } \overline{G}} \arrow[d, Rightarrow, "\textsc{MaxClique}"] \\
\boxed{\underset{k}{\cancel{\text{largest}}}\text{ independent set}} & & \boxed{\underset{k}{\cancel{\text{largest}}}\text{ clique}} \arrow[ll, "\text{trivial}"']
\end{tikzcd}$$



Next: Vertex Cover:
A set of vertices that touches every edge

k-Vertex Cover: Does G have a vertex cover of size k?

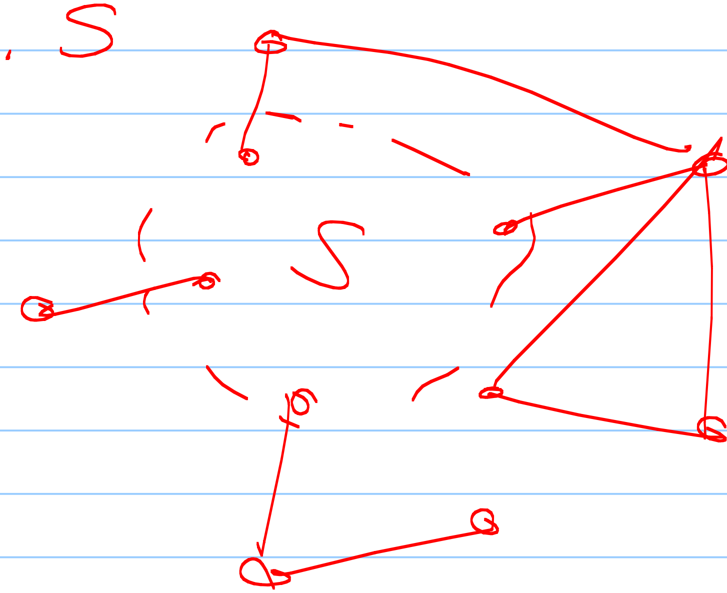
In NP:
- Given a vertex set of size k
- Make a list of edges. $O(n^2)$
- Check that each edge has endpoint in the set.

$\Rightarrow O(n^2 k)$

NP-Hardness: reduce what to this?

Ind Set, S

Key: V−S is a vertex cover, since no edges "inside" S



Given G & K, answer: is the ind. set of size k?

Ask: is there a vertex cover of size n−k?

$$
\begin{tikzcd}
 & \boxed{\text{graph } G = (V, E)} \arrow[r, dashed, "\text{trivial}"] & \boxed{\text{graph } G = (V, E)} \arrow[d, Rightarrow, "\textsc{MinVertexCover}"] \\
\boxed{\cancel{\text{largest}}\text{ independent set } V \setminus S} \arrow[rr, phantom, ""] & & \boxed{\cancel{\text{smallest}}\text{ vertex cover } S} \arrow[ll, dashed, "O(n)"']
\end{tikzcd}
$$

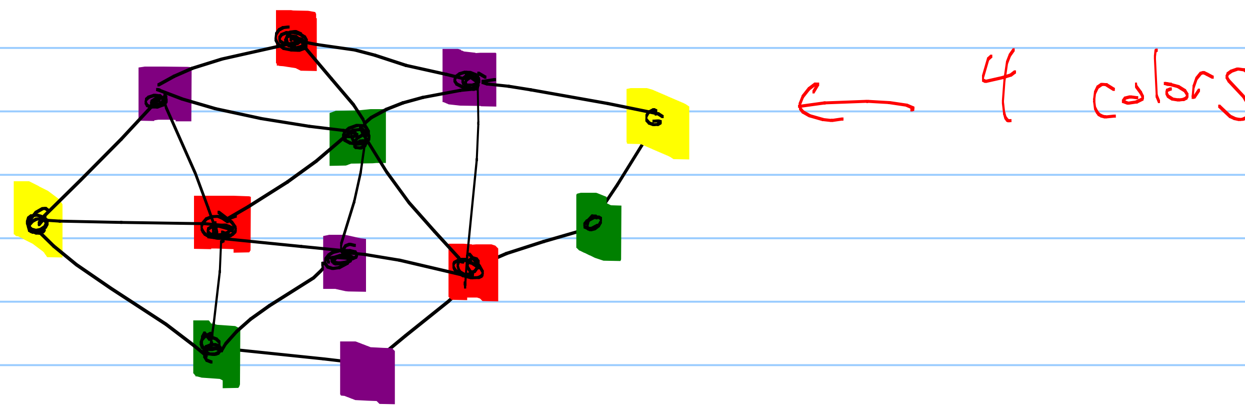
size k

size n-k

Next: Graph Coloring

A *k-coloring* of a graph G is a map
$c: V \to \{1, \ldots, k\}$
that assigns one of k "colors" to each vertex so that every edge has two different colors at endpoints.

Goal: use few colors.



2-coloring? polynomial

Decision version: 3-Colorable: Can G be 3-colored?

In NP: Given a coloring, check no edge has same color at both endpoints.

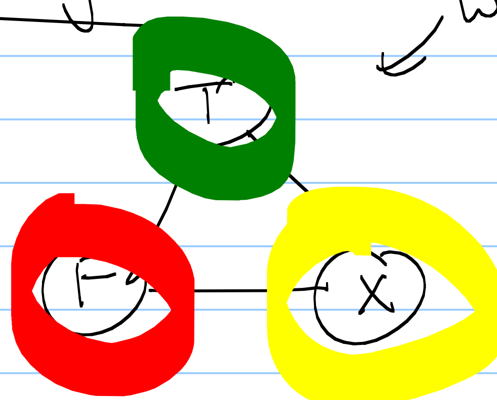
$O(n^2)$

NP-Hard: Reduce 3-SAT.

Given a formula Φ, make G_Φ.

We'll use gadgets, which each incorporate bits of the clause.

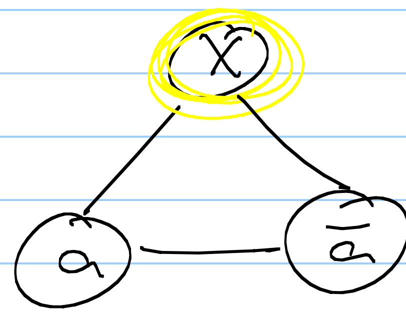
① Truth gadget



will use 3 colors, since all edges present.

(2) Variable gadget:

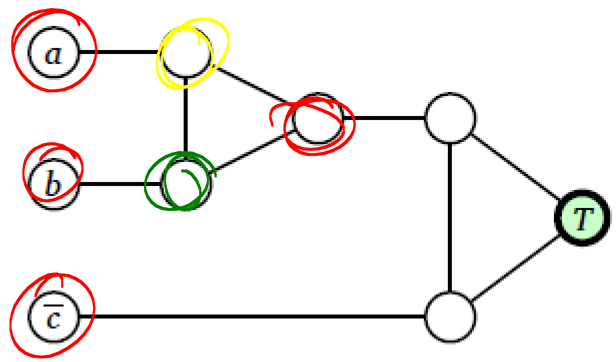
Also – for each variable, make a $\triangle$ with vertex X:



both a & $\bar{a}$ can't be colored "true"

(So a & $\bar{a}$ are set to T/F or F/T, by coloring.)

③ Clause gadget: joins 3 of the variable vertices together to the T vertex:



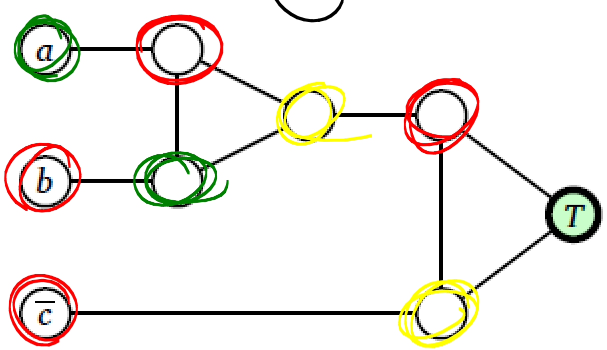
A clause gadget for $(a \vee b \vee \bar{c})$.

T vertex from original △

If all variables are colored false, can't 3-color

(some cases:...)

3-coloring $\Longleftrightarrow$ satisfiable:

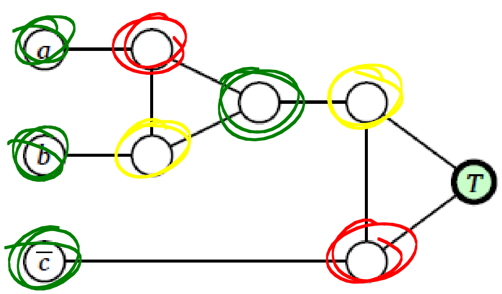


A clause gadget for $(a \vee b \vee \bar{c})$.

$\Longleftarrow$: if ϕ satisfiable, then each clause has "true" variable

$\hookrightarrow$ valid 3-coloring

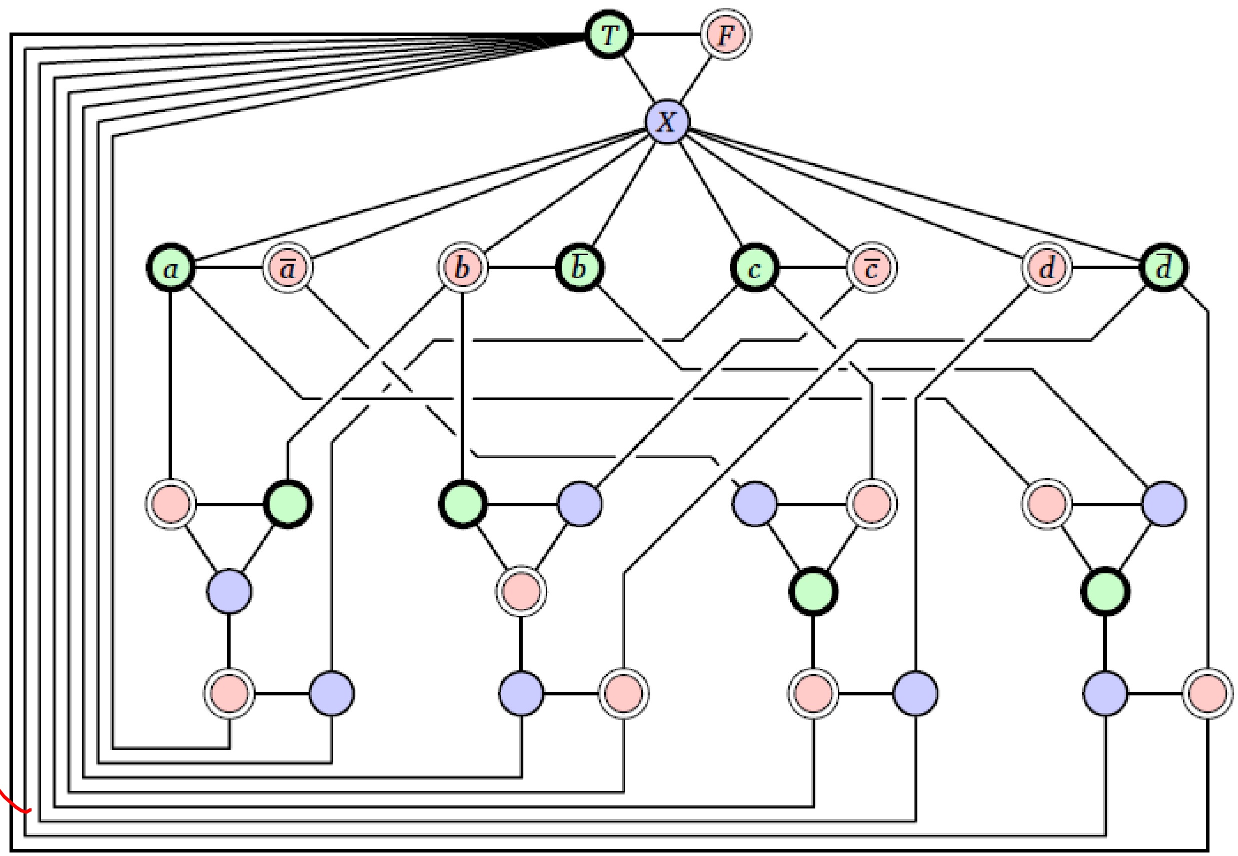
$\Longrightarrow$: if 3-coloring, gives satisfying assignment.



A clause gadget for $(a \vee b \vee \bar{c})$.

Picture:

$a \vee b \vee c$



So:

$$\begin{tikzcd} \boxed{\text{3CNF formula}} \arrow[r, "O(n)"] & \boxed{\text{graph}} \arrow[d, Rightarrow, "\textsc{3Colorable}"] \\ \boxed{\textsc{True or False}} & \boxed{\textsc{True or False}} \arrow[l, "\text{trivial}"'] \end{tikzcd}$$