CS314 - More LP

Note Title

11/20/2013

- HW due Friday

Connections to other problems:

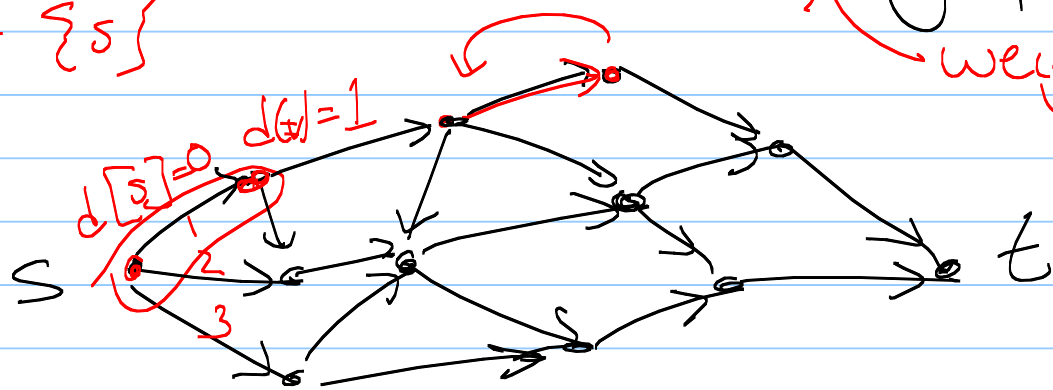
It turns out that LPs are powerful enough to express many other problems.

In a sense, we can reduce many of our other problems to LP:

Ex: Shortest Paths

Goal: find shortest path from s to t in a directed graph G
weighted

$S = \{s\}$



How did our algorithm(s) work?

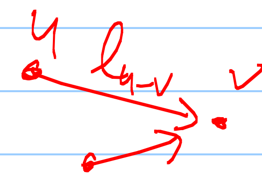
Set up for LP:

variable for each vertex: d_v

maximize d_t

s.t.

$$d_s = 0$$



for each edge $u \to v$, $d_v - d_u \leq l_{u \to v}$

In any feasible solution, d_v is at most the shortest path distance to v.

(Alternative ways to set this up)

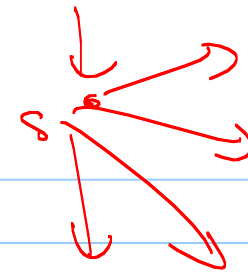
Ex: Flows & Cuts

Input: directed, weighted $G=(V,E)$ & $s,t \in V$

Goal: Choose flow $f: E \to \mathbb{R}$ s.t.

① $0 \le f(e) \le c(e)$

② $\forall v \neq s,t$: $\underbrace{\sum_u f(u \to v)}_{\text{flow into } v} = \underbrace{\sum_w f(v \to w)}_{\text{flow out of } v}$



LP setup:

Maximize flow out of s:

find $f_{u\to w}$

max $\sum_w f_{s\to w} - \sum_v f_{v\to s}$

such that:

for every $u \to v$, $f_{u\to u} \le c_{u\to v}$

for every $v \neq s,t$, $\sum_w f_{v\to w} - \sum_u f_{u\to v} = 0$

$f_{u\to v} \ge 0$

Related: cuts

Let's use indicator variables:

$S_v = 0$ or 1 ← if v is with s

if v is not in S's component

$X_{u \to v} = 1$ if $u \in S$ and $v \in T$

LP :

Note:

For all of these, a solution to the original problem would yield LP solution.

But LP might give wierd fractional result (that) doesn't correspond to a valid cut or path.

Turns out it is possible, but proof is a bit beyond our scope this semester.

Duality:

Recall the reading's example with chocolate:

two types of chocolate, profit \$1 & \$6

LP: $\max \; x_1 + 6x_2$

s.t.

$$\begin{aligned} x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution (last time): $x_1 = 100$, $x_2 = 300$

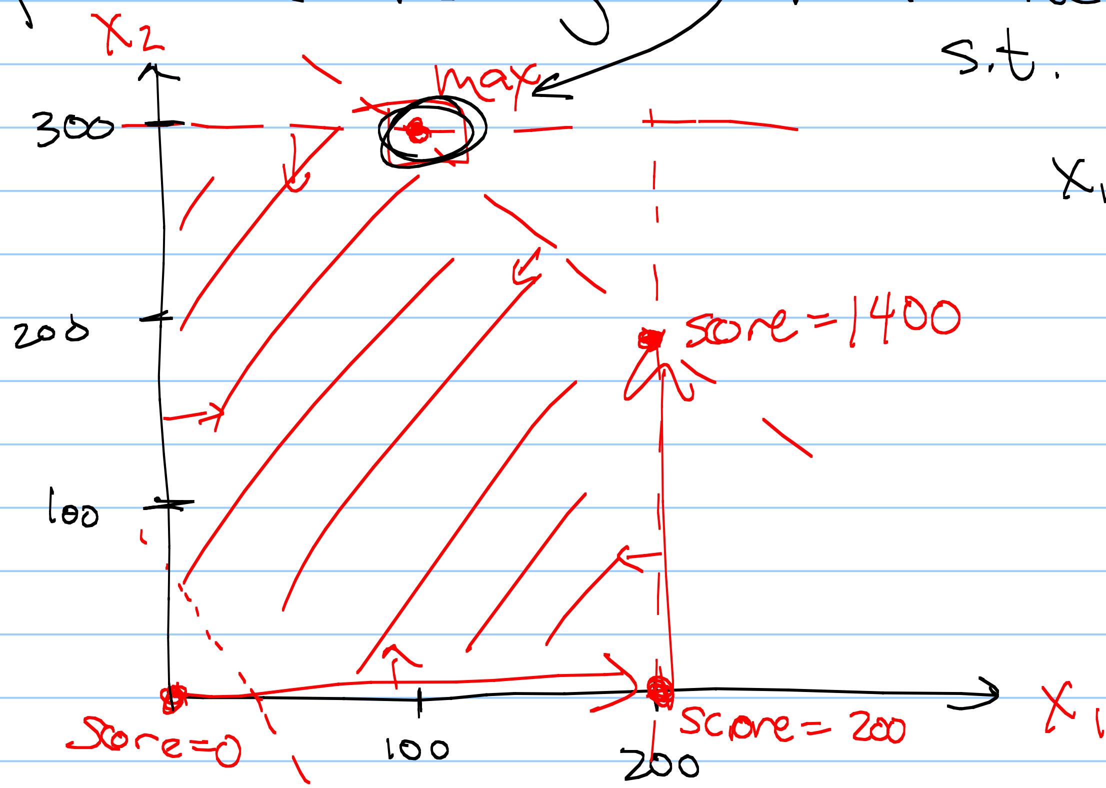
Example from reading: maximize $x_1 + 6 \cdot x_2$

s.t. $x_1 \le 200$

$x_2 \le 300$

$x_1 + x_2 \le 400$

$x_1, x_2 \ge 0$



Can we check this is best somehow?

$$\begin{aligned} \max \quad & x_1 + 6x_2 \\ \text{s.t.} \quad & x_1 \le 200 && (1) \\ & x_2 \le 300 && (2) \\ & x_1 + x_2 \le 400 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Play with inequalities: $(1) + 6 \cdot (2)$

$$\begin{aligned} & x_1 \le 200 \\ + \; & 6x_2 \le 1800 \end{aligned}$$

$\Rightarrow \quad x_1 + 6x_2 \le 2000$

can't hope to do better than \$2000

Interesting! These 2 tell us that we can't do better than \$2000.

But can we get a better combo showing \$1900?

$$\max \quad x_1 + 6x_2$$

s.t.

$$x_1 \le 200 \quad (1)$$
$$x_2 \le 300 \quad (2)$$
$$x_1 + x_2 \le 400 \quad (3)$$
$$x_1, x_2 \ge 0$$

Play: $\underbrace{0 \cdot (1)}_{\text{no}} + 5 \cdot (2) + 1 \cdot (3)$

$$5x_2 \le 1500$$
$$x_1 + x_2 \le 400$$

$\downarrow$ add them

$$x_1 + 6x_2 \le 1900$$

These multipliers $(0, 5, 1)$ are a <u>certificate</u> of optimality,

since no valid solution can possibly do better than \$1900.

How to find these magic values??

Well, 3 "$\leq$" inequalities, so 3 multipliers — y_1, y_2, & y_3

mult for ① ② ③

Multiplier	Inequality	
y_1	$x_1 \leq 200$	$\Rightarrow y_1 \cdot x_1 \leq y_1 \cdot 200$
y_2	$x_2 \leq 300$	$\Rightarrow y_2 \cdot x_2 \leq y_2 \cdot 300$
y_3	$x_1 + x_2 \leq 400$	$\Rightarrow y_3(x_1+x_2) \leq y_3 \cdot 400$

Combine (Add): $(y_1+y_3)x_1 + (y_2+y_3)x_2 \leq \underbrace{200y_1 + 300y_2 + 400y_3}$

Note: Left should look like original certificate so that right is upper bound. So: $(y_1+y_3)x_1 + (y_2+y_3)x_2 = x_1 + 6x_2$

$\Longrightarrow y_1 + y_3 \geq 1$ & $y_2 + y_3 \geq 6$

Goal: A bound as tight as possible

$\Rightarrow$ a new linear program!

Dual LP: minimize: $200y_1 + 300y_2 + 400y_3$

s.t. $y_1 + y_2 \geq 1$

$y_2 + y_3 \geq 6$

$y_1, y_2, y_3 \geq 0$

So if we can find a pair of primal + dual feasible values that match, they are both optimal!

(This is like max flow / min cut, in a way.)

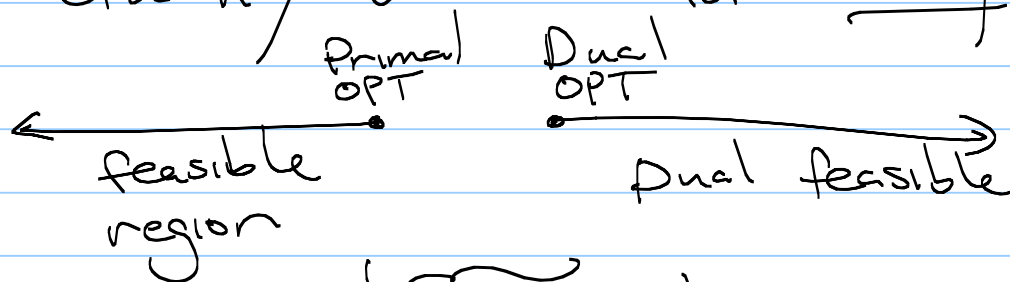
Here: Primal: $(x_1, x_2) = (100, 300)$

matches

Dual: $(y_1, y_2, y_3) = (0, 5, 1)$

Amazing!

This actually works for any LP.



the duality gap is 0.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

In general:

Primal LP:

$$\max \; c^T x$$
s.t.
$$Ax \le b$$
$$x \ge 0$$

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}^T \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Dual LP:

$$\min \; y^T b$$
s.t.
$$y^T A \ge c^T$$
$$y \ge 0$$