CS 314 — Linear Programming

Note Title 11/18/2013

Announcements

- Quiz over reading today
- HW due ~~Wednesday~~ Friday

Dfn: Linear program

A set of variables along with linear equations or inequalities.

Goal: Satisfy these equations/inequalities while also maximizing (or minimizing) some objective function.

Ex: Quiz problem!

Let: w = # wheat] plant

r = # rye] acres

$w + r \geq 7$ $\qquad$ $w, r \geq 0$

$w + r \leq 10$

$200 \cdot w + 100 \cdot r \leq 1200$ ← (amount of \$)

$w + 2 \cdot r \leq 12$ ← (hours of time)

Goal: $500w + 300r$ (maximize)

Example: Diet Problem

n foods, m nutrients

Let $a_{i,j}$ = amount of nutrient i in food j

r_i = requirement of nutrient i

x_j = amount of food j purchased

c_j = cost of food j

Want to satisfy nutrient requirement while minimizing cost.

$A \cdot x$

$$\begin{cases} a_{1,1} \cdot x_1 + a_{1,2} \cdot x_2 + \cdots + a_{1,n} \cdot x_n \geq r_1 \\ a_{2,1} \cdot x_1 + a_{2,2} \cdot x_2 + \cdots + a_{2,n} \cdot x_n \geq r_2 \\ \vdots \\ a_{m,1} x_m + \cdots + a_{m,n} x_n \geq r_m \end{cases}, \quad x_i \geq 0$$

Goal: minimize $c_1 \cdot x_1 + c_2 x_2 + \cdots + c_n x_n$

$$(c_1, c_2, \ldots, c_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Rewrite:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & & \vdots \\ a_{m,1} & \cdots & \cdots & a_{m,n} \end{pmatrix}$$

$$r = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \quad c = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}$$

$\boxed{\text{Why?}}$

Rewrite:

minimize: $c \cdot x$

s.t.

$$A \cdot x \geq r$$

$$x \geq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

In general:

$$\text{maximize} \sum_{j=1}^{d} c_j x_j$$ (or minimize)

$$\begin{aligned}
\text{subject to} \sum_{j=1}^{d} a_{ij} x_j &\leq b_i \quad \text{for each } i = 1 \,..\, p \\
\sum_{j=1}^{d} a_{ij} x_j &= b_i \quad \text{for each } i = p+1 \,..\, p+q \\
\sum_{j=1}^{d} a_{ij} x_j &\geq b_i \quad \text{for each } i = p+q+1 \,..\, n
\end{aligned}$$

Canonical form:

$$\text{maximize} \sum_{j=1}^{d} c_j x_j$$

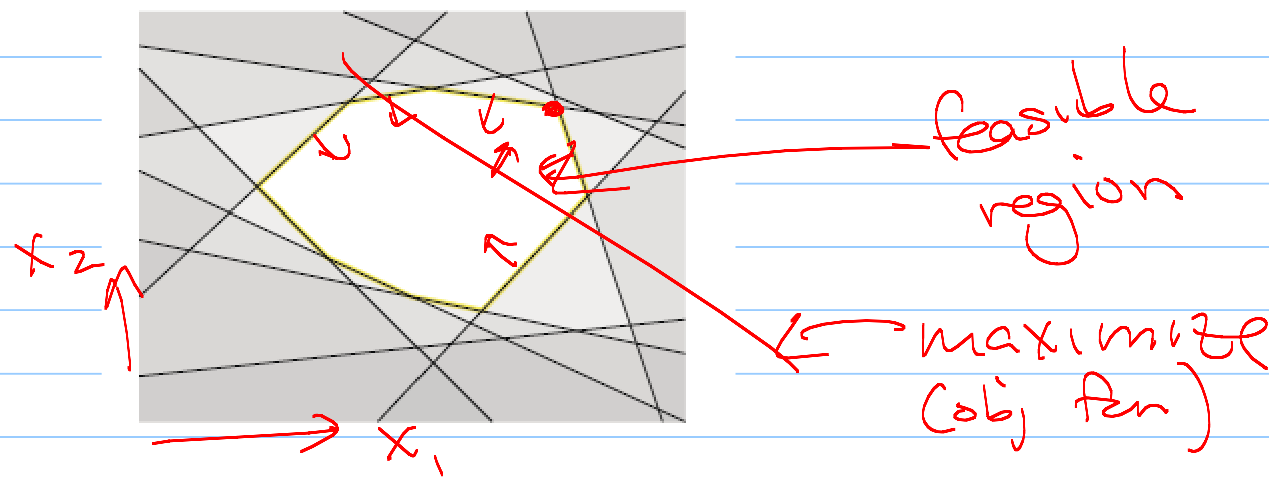
$$\text{subject to} \sum_{j=1}^{d} a_{ij} x_j \le b_i \quad \text{for each } i = 1 \,..\, n$$

$$x_j \ge 0 \quad \text{for each } j = 1 \,..\, d$$

(more like inital example)

Geometric picture:

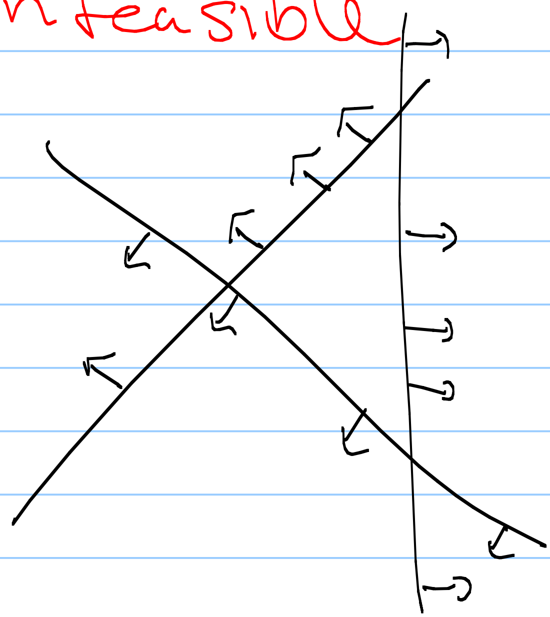
10 inequalities + 2 variables x_1, x_2:



(Higher dimensions harder to visualize...)

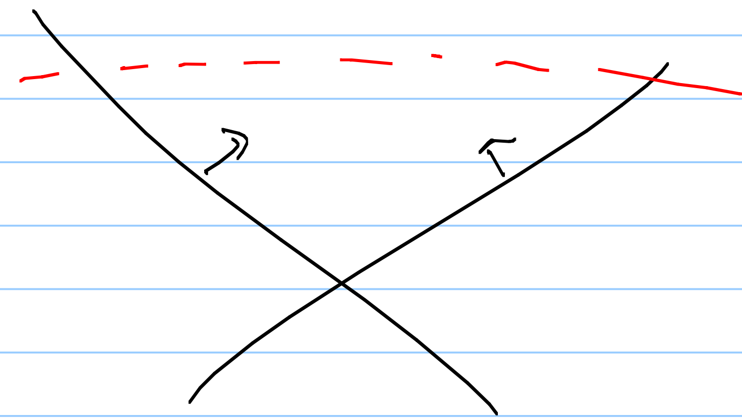
How could it be not solvable?
(2 possibilities)

Infeasible



or

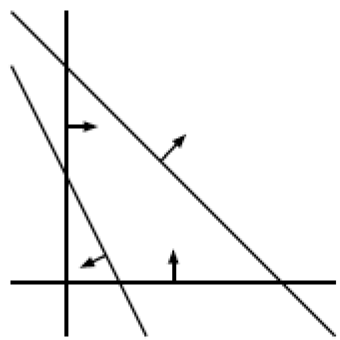
unbounded



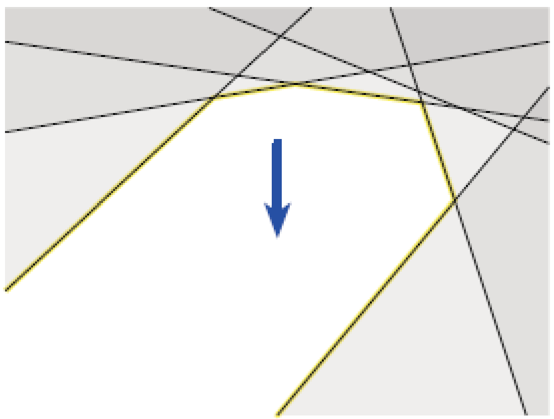
(better pictures)

infeasible

$$\begin{aligned} &\text{maximize } x - y \\ &\text{subject to } 2x + y \leq 1 \\ &\qquad x + y \geq 2 \\ &\qquad x, y \geq 0 \end{aligned}$$



unbounded



Notes:

$x \geq y \iff -x \leq -y$

- Multiplying by -1 turns maximization into minimization

- To turn inequality into equality:

$$\sum_{i=1}^{n} a_i x_i \leq b$$

$$\Downarrow \quad s \text{ be slack variable}$$

$$\sum_{i=1}^{n} a_i x_i + s = b, \quad \text{bound } s \text{ if needed}$$

- To change equalities to inequalities:

$$\sum_{i=1}^{n} a_i x_i = b$$

Rewrite: 2 equations

$$\sum a_i x_i \geq b \quad \& \quad \sum a_i x_i \leq b$$

Solving linear programs: Simplex

Example from reading:

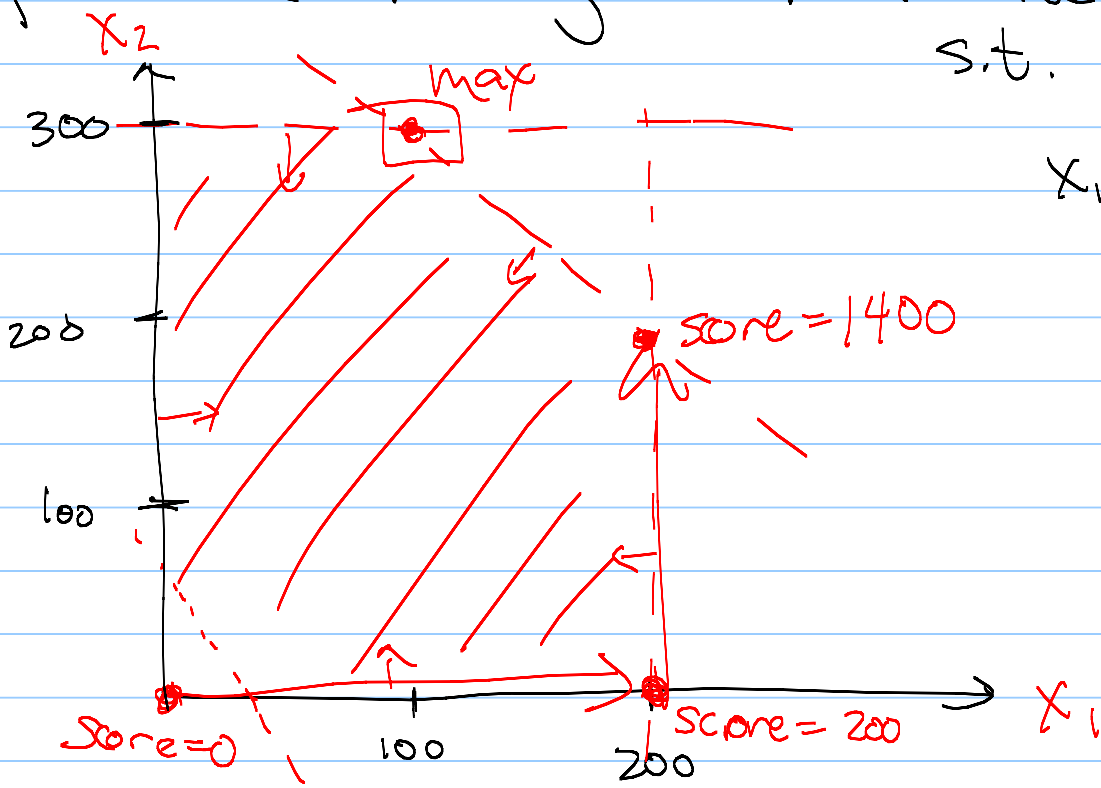
maximize $x_1 + 6 \cdot x_2$

s.t. $x_1 \le 200$

$x_2 \le 300$

$x_1 + x_2 \le 400$

$x_1, x_2 \ge 0$

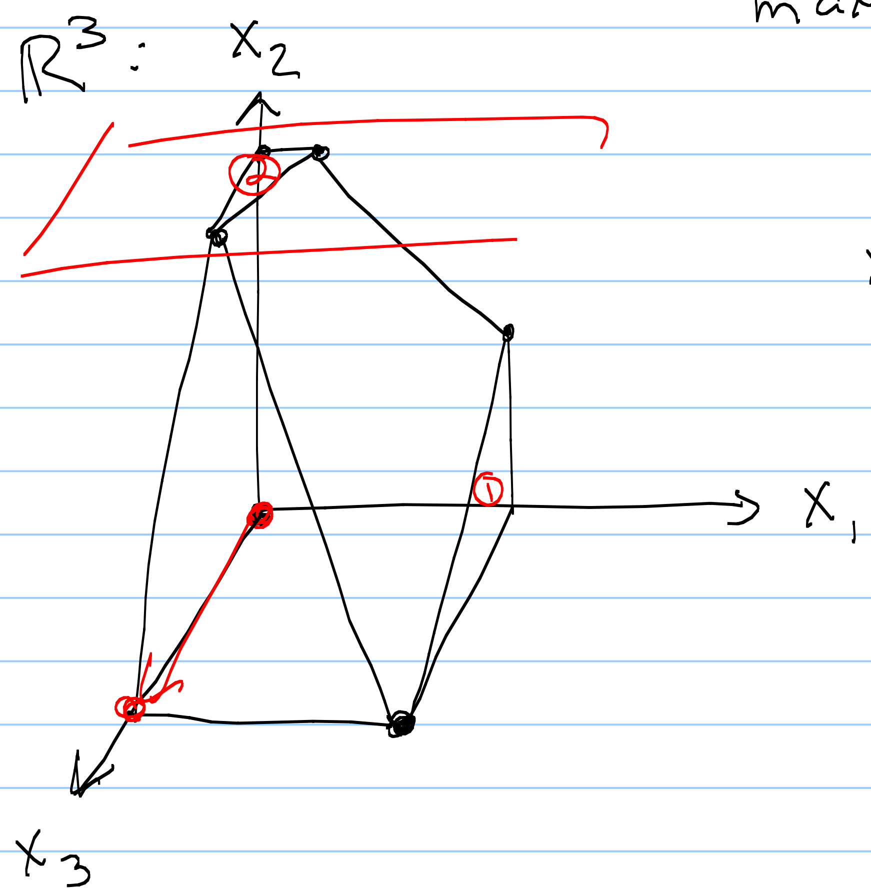


Simplex: start at a vertex v of the solution polyhedron.

While v has a better neighbor
set v = this better neighbor.

If no better neighbor, done.
Why? (in 2-d)

In $\mathbb{R}^3$:



maximize $x_1 + 6x_2 + 13x_3$

$x_1 \le 200$ 1

$x_2 \le 300$ 2

$x_1 + x_2 + x_3 \le 400$

$x_2 + 3x_3 \le 600$

$x_1 \ge 0$

$x_2 \ge 0$

$x_3 \ge 0$

In $\mathbb{R}^n$: hyper planes which describe the boundary

So what is a neighbor?