Math 135 – Solving Linear Recurrences

Note Title 10/29/2012

Recap: Solving linear homogeneous recurrences

1) Find characteristic eqn (& roots).

2) If r is a non-repeated root of the char. eqn, then r^n is a solution to the recurrence

3) If r is a <u>repeated</u> root with multiplicity k, then:

$$r^n, n \cdot r^n, n^2 \cdot r^n, \dots, n^{k-1} r^n$$

are all solutions

4) Solve using base case(s) with linear combinations of 2) & 3)

Ex: $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 2$, $a_1 = 7$

degree 2

char eqn: $x^2 = x + 2$

$\Rightarrow x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

roots: $2, -1$

$a_n = c_1(2)^n + c_2(-1)^n$

$a_0 = 2 = c_1 \cdot 2^0 + c_2 \cdot (-1)^0 = c_1 + c_2$ ① $\Rightarrow c_1 = 2 - c_2$

$a_1 = 7 = c_1 \cdot 2^1 + c_2(-1)^1 = 2c_1 - c_2$

② $\hookrightarrow 7 = 2(2 - c_2) - c_2$

$7 = 4 - 3c_2 \Rightarrow c_2 = -1$

$c_1 + (-1) = 2 \Rightarrow c_1 = 3$

Answer:

$a_n = 3 \cdot 2^n + (-1) \cdot (-1)^n$

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

<u>Ex</u>: $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$

$x^2 - x - 1 = 0$

$x = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$ (last time)

$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$

solve for c_1 + c_2:

$F_0 = 0 = c_1 + c_2 \Rightarrow c_1 = -c_2$

$F_1 = 1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) \Rightarrow c_1 = \frac{1}{\sqrt{5}}$

$1 = -c_2\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$

$1 = c_2\left(\frac{-1-\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}\right) \quad c_2 = \frac{1}{\frac{-\cancel{2}\sqrt{5}}{\cancel{2}}} = \frac{1}{-\sqrt{5}}$

Dfn: Linear ~~inhomogeneous~~ recurrences have an added function $g(n)$:

$$f(n) = c_1 f(n-1) + \cdots + c_d f(n-d) + g(n)$$

Ex: $F_n = F_{n-1} + F_{n-2} + 1$ ← (poly of deg 0) $\cdot 1^n$

$A(n) = 4A(n-1) + 3^n$

(poly of deg 0) $\cdot 3^n$

Method:

1. "Ignore" g(n) + find general solution for the homogeneous part
2. Find general solution for g(n) complex (more on next slide)
3. Add them together
4. Solve for constants using base cases (+ possibly the recurrence)

Step 2: can be complex

We'll talk about how to solve for $g(n)$ when it is of the form:

$$g(n) = (\text{polymomial of degree } k) \cdot s^n$$

where s is constant.

Ex:

$$g(n) = \underbrace{(n^2+1)}_{\text{poly of deg 2}} \cdot \underbrace{2^n}_{2^n}$$

$$g(n) = \underbrace{(2n-5)}_{\text{poly of deg 1}} \cdot 1^n$$

How to solve:

Is s a char root?

No → try general solution of the form (poly of deg k) $\cdot s^n$

Yes → what is its multiplicity?

$\xrightarrow{m}$ try general solution of the form n^m (poly deg k) $\cdot s^n$

Ex: $f(0)=1$

$f(n)=4f(n-1)+3^n$

1 Ignore $g(n)=3^n$

have deg 1 : $x-4=0$

so root is 4, guess $c_1 \cdot 4^n$

2 $g(n)=3^n$

not a char root

try : $c_2 \cdot 3^n$

3-4 $f(n)=c_1 \cdot 4^n + c_2 \cdot 3^n$ ←

$f(0)=1 = c_1+c_2$

$f(1)=4\cdot 1+3^1=7=4c_1+3c_2$

$$f(n) = c_1 \cdot 4^n + c_2 \cdot 3^n \leftarrow$$

① $f(0) = 1 = c_1 + c_2$

② $f(1) = 4 \cdot 1 + 3^1 = 7 = 4c_1 + 3c_2$

① $\Rightarrow c_1 = 1 - c_2$

② $7 = 4(1 - c_2) + 3c_2$

$7 = 4 - c_2 \Rightarrow c_2 = -3$

$c_1 = 4$

Ans: $\boxed{f(n) = 4 \cdot 4^n - 3 \cdot 3^n}$

Ex: $a_n = 5a_{n-1} - 6a_{n-2} + \underbrace{(n^2-n)\cdot 7^n}_{g(n)}$ → give general form

① $a_n = 5a_{n-1} - 6a_{n-2}$

$x^2 = 5x - 6$

$x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0$

roots: 2, 3

form $c_1 \cdot 2^n + c_2 \cdot 3^n$

② $g(n) = \underbrace{(n^2-n)}_{\text{deg } 2} \cdot 7^n$ ← $s = 7$

gen form: $(c_3 \cdot n^2 + c_4 \cdot n + c_5) \cdot 7^n$

Final general solution: $a_n = c_1 \cdot 2^n + c_2 \cdot 3^n + (c_3 n^2 + c_4 n + c_5) \cdot 7^n$

$$g(n)$$

Ex: $a_n = 6a_{n-1} - 9a_{n-2} + n \cdot 3^n$

① $x^2 - 6x + 9 = 0$ (char eqn)

$(x-3)(x-3) = 0$

root: 3, mult: 2

poly of deg. mult − 1

soln: $c_1 3^n + c_2 n 3^n = (c_1 + c_2 n)3^n$

② $g(n) = n \cdot 3^n$ ← $s = 3$

poly of deg 1

$n^m (\text{poly deg } k) \cdot s^n$

gen form: $n^2 (c_3 n + c_4) 3^n$

General solution: $a_n = (c_1 + c_2 n) \cdot 3^n + n^2 (c_3 n + c_4) 3^n$

Ex: $a_n = 6a_{n-1} - 9a_{n-2} + n^3 - 4$

(1) $x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$

root: 3 w/ deg 2

gen form: $(c_1 n + c_2) \cdot 3^n$

(2) $g(n) = \underbrace{(n^3 - 4)}_{\text{deg } 3} \underbrace{1^n}_{s=1}$

gen: $(c_3 n^3 + c_4 n^2 + c_5 n + c_6) \cdot 1^n$

Final: $a_n = (c_1 n + c_2) 3^n + (c_3 n^3 + c_4 n^2 + c_5 n + c_6)$

Ex: $a_n = a_{n-1} + n$, $a_0 = 0$

① char eqn: $x = 1$

$x - 1 = 0$

root: 1

gen form: $c_1 \cdot 1^n$

② $g(n) = (n)\, 1^n$

deg 1, $s = 1$

gen form: $n^1 (c_2 n + c_3) \cdot 1^n$

$a_n = c_1 \cdot 1^n + n(c_2 n + c_3) \cdot 1^n$

$= c_1 + c_2 \cdot n^2 + c_3 \cdot n$

$$a_n = a_{n-1} + n \;, \quad a_0 = 0$$

$$a_n = c_1 \cdot 1^n + n(c_2 n + c_3) \cdot 1^n$$
$$= c_1 + c_2 \cdot n^2 + c_3 \cdot n$$

$$a_0 = 0 = c_1 + c_2 \cdot 0^2 + c_3 \cdot 0 \Rightarrow c_1 = 0$$
$$a_1 = 1 = c_2 \cdot 1^2 + c_3 \cdot 1 = c_2 + c_3$$
$$a_2 = 3 = c_2 \cdot 4 + c_3 \cdot 2 = 4c_2 + 2c_3$$

$$c_2 = 1 - c_3$$
$$3 = 4(1 - c_3) + 2c_3$$
$$3 = 4 - 4c_3 + 2c_3$$
$$-1 = -2c_3$$
$$c_3 = \frac{1}{2} \qquad c_2 = \frac{1}{2}$$

$$\boxed{a_n = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{n(n+1)}{2}}$$

Divide and Conquer Recurrences

Non-Linear, but in terms of smaller values in the sequence based on division; eg:

$$\boxed{f(n) = a f\left(\frac{n}{b}\right) + g(n)}$$

Ex:

$$B(n) = B\left(\frac{n}{2}\right) + 1$$

$$M(n) = 2M\left(\frac{n}{2}\right) + n$$

$$f(n) = 7 f\left(\frac{n}{2}\right) + \frac{15n^2}{4}$$

Unrolling:

Ex: $B(n) = B(\frac{n}{2}) + 1$, $B(1) = 1$