Math 135 - Set Theory

Note Title 9/10/2012

Announcements

- HW2 up - due Friday
  (Cover sections 1.7 & 1.8)

- Thursday - move office hours
  to 9-10 am

Sets (2.1)

Definition: A set is an unordered collection of objects.

Ex:

$\phi = \{ \}$ empty set

$\{1, 3, 5, 7\}$

$\{1, 2, 3, 4, \ldots, 100\}$

$\{a, b, c, \ldots, z\}$

$\{\phi, \{1\}, \{1, 3\}\}$

Definitions

- A set is said to contain its <u>elements</u> (or <u>members</u>)

- Two sets are <u>equal</u> if & only if they contain the same elements

<u>Ex</u>: $\{1, 3, 5\} = \{5, 3, 1\}$

good

$= \{1, 3, 1, 1, 5, 3\}$

repetition & order don't matter

Examples:

Natural Numbers : $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

Integers : $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Rationals : $\mathbb{Q} = \{ p/q : p \in \mathbb{Z}, q \in \mathbb{Z}, \text{ and } q \neq 0 \}$

Real Numbers : $\mathbb{R}$

such that

Ways to define a set

- List: $S = \{1, 2, 3, 4\}$.

  $T = \{0, 1, 4, 9, 16, \ldots\}$ ← squares

- English: "Let T be the set of squares."
- Form description: $T = \{n^2 : n \in \mathbb{N}\}$
- Property description: $T = \{n \in \mathbb{N} : n \text{ is a perfect square}\}$

$X = \{3x+2 : x \in T\}$

Notation:

- $x \in S$ means x is a member of S
- $x \notin S$ means x is not a member of S
- $A \subseteq B$ means A is a subset of B

  → Formally: $\forall x,\ x \in A \rightarrow x \in B$

  Note: $A = B \longleftrightarrow (A \subseteq B \wedge B \subseteq A)$
- $A \subsetneq B$ or $A \subset B$ means A is a proper subset of B

  so $A \subseteq B$ and $A \neq B$

Examples:

$\mathbb{N} \subset \mathbb{Z}$: $-1 \in \mathbb{Z}$, $-1 \notin \mathbb{N}$

$\sqrt{5} \in \mathbb{R}$

$\sqrt{2} \notin \mathbb{Q}$ ⟵ pf by contradiction

$2 \in \{1, 2, 3\}$

Also: $S \subseteq S$

<u>Lemma</u>: For any set S, $\phi \subseteq S$.

<u>pf</u>: For any x, if $x \in \phi$, then $x \in S$.

Take any x.

$x \notin \phi$.

(this is vacuously true) $\boxtimes$

<u>Note</u>: <u>Not</u> saying $\phi \in S$.

$\phi \subset \{1, 2, 3\}$

$\phi \notin \{1, 2, 3\}$

$\phi \in \{\{1\}, \phi, 2\}$

$\phi \subset$?

Sets: more definitions

Let S be a set.
If S has exactly n (unique) elements, then we say S is <u>finite</u>, with <u>cardinality</u> n.

<u>Notation</u>: $|S| = n$

S is said to be <u>infinite</u> if it is not finite.

Infinite sets?

$\mathbb{R}, \mathbb{Z}, \mathbb{Q}, \mathbb{N}$

Power Set

The power set of S, written $P(S)$ or 2^S, is the set of all subsets of S.

Ex: Let $S = \{0, 1, 2\}$.

$P(S) = \{ \phi, \{0\}, \{1\}, \{2\},$
$\{1,2\}, \{0,2\}, \{0,1\},$
$\{0,1,2\} \}$

Ex: Let $S = \{a, 1, \sqrt{2}\}$. What is $P(S)$?

$\{\phi, \{a\}, \{1\}, \{\sqrt{2}\}, \{a,1\}, \{1,\sqrt{2}\},$
$\{a,\sqrt{2}\}, \{a,1,\sqrt{2}\}\}$

Ex: What is $P(\phi)$? $S = \phi$

$\{\phi\} = \{\{\}\}$

$P(\{\phi\}) = \{\phi, \{\phi\}\}$

$\exists x \in S$ s.t. $1 \in x$

Let $S = \{\emptyset, \{1,2\}, \sqrt{2}\}$.

$P(S) = \{\emptyset, \{\emptyset\}, \{\{1,2\}\}, \{\sqrt{2}\},$
$\{\emptyset, \{1,2\}\}, \{\{1,2\}, \sqrt{2}\},$
$\{\emptyset, \sqrt{2}\}, \{\emptyset, \{1,2\}, \sqrt{2}\}\}$

$1 \notin S$

$\emptyset \in S$ and $\emptyset \in P(S)$

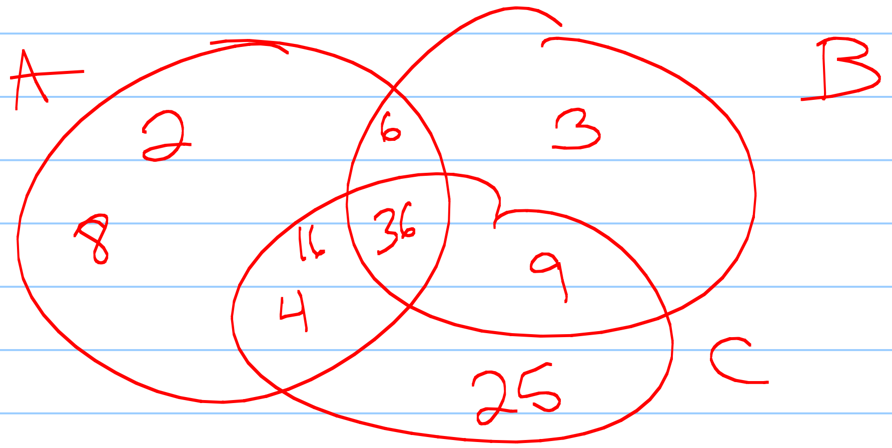
$\{1,2\} \in S$

$\{1,2\} \not\subseteq S$

Venn Diagrams ⟵⟶ not a proof!

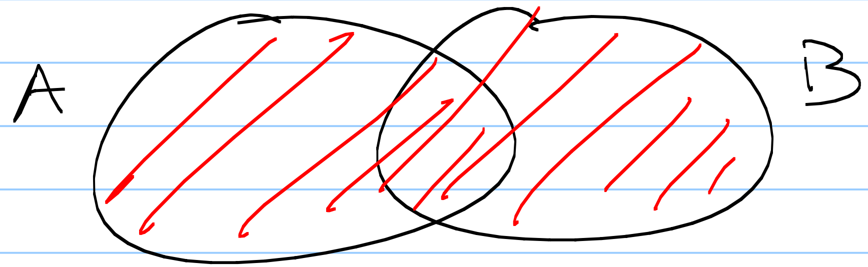
Sometimes we want a picture of how sets interact.

Ex:
$A = \{n \in \mathbb{N} : n \text{ is even}\}$ 2, 4, 6, 8, ...
$B = \{n \in \mathbb{N} : n \text{ is divisible by } 3\}$ 3, 6, 9, 12, ...
$C = \{n^2 : n \in \mathbb{N}\}$ 1, 4, 9, 16, 25 ...

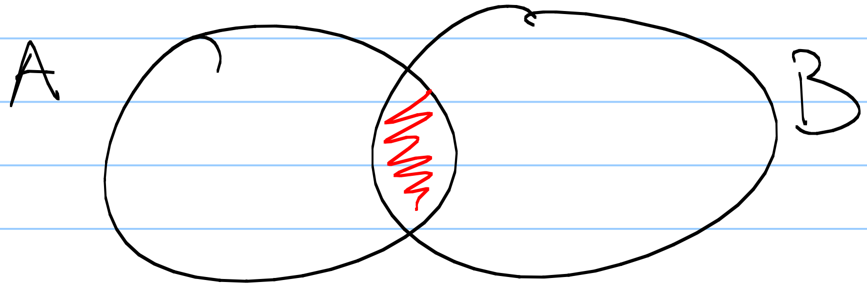


Definitions

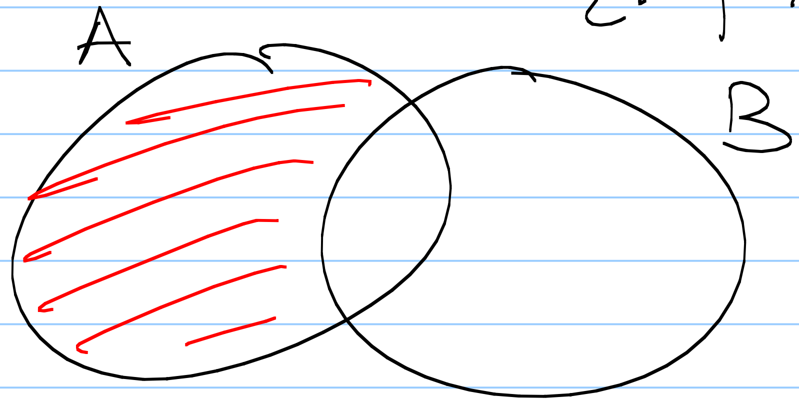
Union: $A \cup B = \{x \mid x \in A \lor x \in B\}$



Intersection: $A \cap B = \{x \mid x \in A \land x \in B\}$



Set difference: $A - B = \overbrace{\{x \mid x \in A \wedge x \notin B\}}$



$A - B \neq B - A$

Dfn: Two sets are called **disjoint** if their intersection is empty, so $A \cap B = \emptyset$.

Examples

$$A = \{2, 7, \{a,b\}, \pi\}$$
$$B = \{\sqrt{2}, \pi, a, b\}$$
$$C = \{\{a\}, b, \{a,b\}\}$$

$$A \cup B = \{2, 7, \{a,b\}, \pi, \sqrt{2}, a, b\}$$

$$A \cap B = \{\pi\}$$

$$(A \cap C) \cup B = \{\{a,b\}, \sqrt{2}, \pi, a, b\}$$

$$B - C = \{\sqrt{2}, \pi, a\}$$

$$P(A \cap B) = \{\phi, \{\pi\}\}$$

$$\mathcal{P}(\underset{\text{not a set}}{1})$$