Math 135 – Proofs (cont)

Note Title 9/6/2012

Announcements

- Turn in HW1 now
- HW2 up tonight (on proofs)

Pf by contradiction: (~1.7)

So to show p is true, our method is:

- assume p is false
- derive a contraction

(& then p must be true)

Ex: Prove that $\sqrt{2}$ is irrational.

pf: Suppose $\sqrt{2}$ is rational.

Can write $\sqrt{2} = \frac{p}{q}$ $p, q \in \mathbb{Z}$, & $q \neq 0$.

and p/q is reduced form,

So no common divisors.

$\Rightarrow$ $2 = \frac{p^2}{q^2}$ so $2q^2 = p^2$

so p^2 is even

$\Rightarrow$ p is even (by earlier lemma).

pf cont:

Then $p = 2k$, some $k \in \mathbb{Z}$

Can rewrite $2q^2 = p^2$ as

implies

$$2q^2 = (2k)^2$$

$$\Rightarrow 2q^2 = 2^2k^2 = 4k^2$$

$$\Rightarrow q^2 = 2k^2$$

So q^2 is even!

so q is even (by same prev. ex)

contradiction

So p & q must have common divisor 2

contradicts the fact that p/q can be written in reduced form. $\boxtimes$

2-way implications

$p \to q$
$p \Rightarrow q$ } one way

Ex: Suppose n is an integer.
n is odd $\Longleftrightarrow$ n^2 is odd.
$\longleftrightarrow$

if and only if

Pf: Need 2 proofs! $p \to q$

① If n is odd, then n^2 is odd.

Assume n is odd.
so $n = 2k+1$
so $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$
so n^2 is odd.

2^{nd} way:

$q \to p$

2. If n^2 is odd, then n is odd.

pf by contrapositive:

If n is even, then n^2 is even.

So assume n is even.

$n = 2k$

$n^2 = 4k^2 = 2(2k^2)$

So n^2 is even. ∎

Another cases example:

Prove that $\forall n \in \mathbb{Z}$, $n^2 \geq n$.

Cases: ① Assume n is positive, so $n \geq 1$.

Since n is positive, can multiply both sides by n, & it is still true:

$n \cdot n \geq 1 \cdot n$

$\Rightarrow n^2 \geq n$.

② say $n = 0$, then $n^2 = 0$. (and $0 \geq 0$)

③ $n < 0$: n^2 is positive.

$\Rightarrow n^2 > 0$

Assumed $n < 0$

So $n < 0 < n^2 \Rightarrow n < n^2$. ∎

$(x^a)^b$

Non-constructive proofs:

Show that there exist irrational numbers x and y with x^y rational.

pf: Consider $\sqrt{2}$.

Is $\sqrt{2}^{\sqrt{2}}$ rational?

If so, let $y=\sqrt{2}$ & $x=\sqrt{2}$, & done.

If not, then $\sqrt{2}^{\sqrt{2}}$ is irrational.

Let $x=\sqrt{2}^{\sqrt{2}}$

and $y=\sqrt{2}$,

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = (\sqrt{2})^2$$

$$= 2$$

$\blacksquare$

Another trick:

Prove that the arithmetic mean $\frac{x+y}{2}$ of 2 positive real numbers is always greater than their geometric mean, $\sqrt{xy}$.

Goal: Show $\frac{x+y}{2} > \sqrt{xy}$.

Now the "forwards" proof: