Math 135 - Proofs (1.7?)

Note Title 9/5/2012

Announcements

- HW1 is due Friday

Proofs:

A theorem (or lemma, proposition, etc.) is a statement that can be rigourously shown to be true.

Generally something like:

"If n is even, then it is divisible by 2."

"$\forall \varepsilon > 0 \ \exists \delta > 0$ such that if $|f(x) - f(y)| < \varepsilon$, then $|x - y| < \delta$."

The sequence of statements giving that arguement is called a proof.

Direct proofs

Think about $p \to q$.

When is it false?

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

So to show $p \to q$ is true need to show not in row 2.

A few definitions

n is even $\iff$ $n = 2k$ for some $k \in \mathbb{Z}$ ← even

n is odd $\iff$ $n = 2k+1$ for some $k \in \mathbb{Z}$

Ex: $\overbrace{\text{If } n \text{ is an odd integer}}^{p}$,
then $\underbrace{n^2 \text{ is even}}_{q}$.

true or false ?

give "row 2" example

3 is odd

9 is not even

Ex: If n is an even integer, then n^2 is even.

Proof: (Assume p is true, & show q can't be false.)

Assume n is even.

Then $n = 2k$ for some $k \in \mathbb{Z}$

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

and $2k^2 \in \mathbb{Z}$ (since $k \in \mathbb{Z}$)

so n^2 must be even.

direct proof

Ex: If x is even and y is odd, then $x+y$ is odd.

pf: Assume x is even and y is odd.

So $x = 2k$ and $y = 2\ell + 1$ (by def).
(k and $\ell \in \mathbb{Z}$)

Consider
$$x+y = 2k + 2\ell + 1$$
$$= 2(k+\ell) + 1$$
and $k+\ell \in \mathbb{Z}$

So $x+y$ is odd.

Indirect Proofs

Recall: $p \to q$ is logically equivalent to: $\neg q \to \neg p$

(What is this called?) contrapositive

Since they are equivalent, if $p \to q$ seems difficult, we can instead consider the logically equivalent implication.

Ex: If $3n+2$ is odd, then n is odd.

Try direct:

Assume $3n+2$ is odd

$\Rightarrow \quad 3n+2 = 2k+1, \quad k \in \mathbb{Z}$

so $3n = 2k-1$

$n = \frac{2k-1}{3}$?

Ex: If $\overbrace{3n+2 \text{ is odd}}^{p}$, then $\overbrace{n \text{ is odd}}^{q}$.

Try contrapositive: $\neg q \rightarrow \neg p$

If n is even, then 3n+2 is even.

Pf: Assume n is even,

$\Rightarrow$ $n = 2k$, for some $k \in \mathbb{Z}$.

then $3n+2 = 3(2k)+2$

$= 2(\underbrace{3k+1}_{\in \mathbb{Z}})$

so 3n+2 is even. ∎

$a > \sqrt{n} \Rightarrow ab > \sqrt{n} \cdot b > \sqrt{n} \cdot \sqrt{n}$

Ex: Prove that if $\overbrace{n = ab}^{p}$ (for a, b positive integers), then $\underbrace{a \le \sqrt{n} \text{ or } b \le \sqrt{n}}_{q}$.

proof: contrapositive $\neg q \to \neg p$:

Assume that a not $\le \sqrt{n}$ and b not $\le \sqrt{n}$.

Assume $a > \sqrt{n}$ and $b > \sqrt{n}$.

Since a and b positive,

$$ab > \sqrt{n} \cdot \sqrt{n}$$

$$\underbrace{ab > n}_{\neg p} \Rightarrow n \neq ab$$

so proved contrapositive $\blacksquare$

Using cases

Thm: $\forall n \in \mathbb{Z}$, n^2+n is even.

proof:

Case 1: Say n is even.

$n = (2k)$, $k \in \mathbb{Z}$.

$n^2 = 4k^2$

$n^2+n = 4k^2+2k = 2(\underbrace{2k^2+k}_{\in \mathbb{Z}})$

so even.

Case 2: Spps n is odd.

$n = 2\ell+1$, $\ell \in \mathbb{Z}$

$n^2 = (2\ell+1)^2 = 4\ell^2+4\ell+1$

$n^2+n = (4\ell^2+4\ell+1)+(2\ell+1)$

$= 4\ell^2+6\ell+2$

$= 2(2\ell^2+3\ell+1)$

so n^2+n is even. ∎

$r \in \mathbb{Q}$

Dfn: A number r is rational if $\exists\, p, q \in \mathbb{Z}$ with $q \neq 0$ such that $r = p/q$.

Ex: $10/5$, $3/1$, $5/11$, ...

A real number that is not rational is called irrational.

Ex: π, e, ϕ, $\sqrt{2}$, ...

Dfn: Reduced form: when p and q have no common divisors.

Ex: $2/3$, $9/10$, $11/6$, ...

not $5/10$

Ex: Prove that the sum of 2 rational numbers is irrational.

(How to rewrite as $p \to q$?)

Proof by contradiction

A contradiction is a logical statement which is always false.

Ex: $x = x + 1$

These can be useful in proof techniques: if we make an assumption & then can show a contradiction, our inital assumption must be false!

Why?

Suppose we can show $\neg p \to q$, where q is a contradiction.

$\neg p$	q	$\neg p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

So which row must be our case?

that p is false

Pf by contradiction:

So to show p is true, our method is:

- assume p is false
- derive a contraction

(& then p must be true)

Ex: Prove that $\sqrt{2}$ is irrational.

pf: Suppose $\sqrt{2}$ is rational.

Can write $\sqrt{2} = \frac{p}{q}$ $p, q \in \mathbb{Z}$, & $q \neq 0$.

and p/q is reduced form,

So no common divisors.

$\Rightarrow 2 = \frac{p^2}{q^2}$ so $2q^2 = p^2$

So p^2 is even

$\Rightarrow p$ is even (by earlier lemma).