# Math 135: More Logic

Note Title 8/31/2012

## Announcements

- HW1 will be up after class (due next Friday)

Ex: Truth tellers & liars

Alice: "Exactly one of us is telling the truth."

Bob: "We are all lying."

Cindy: "The other two are lying."

Same idea – bigger table!

$p$ : Alice is truthful

$q$ : Bob is truthful

$r$ : Cindy is truthful.

| | p | q | r | Exactly 1 truthful | All lying | Other 2 lying |
|---|---|---|---|---|---|---|
| X | T | T | T | F | F | F |
| X | T | T | F | F | F | F |
| X | T | F | T | F | F | F |
| ~~X~~ | T | F | F | T | F | F |
| X | F | T | T | F | F | F |
| X | F | T | F | T | F | F |
| X | F | F | T | T | F | T |
| X | F | F | F | F | T | T |

Alice is only honest person.

# Worksheet example:

| Kevin | Dan | "Dan is lying" | "Both telling truth" |
| --- | --- | --- | --- |
| T | T | F | T |
| T | F | T | F |
| F | T | F | F |
| F | F | T | F |

Paradox:

It is possible to have no consistent rows - this is known as a paradox.

It is also possible to have multiple possible rows - in this case, can't decide who is truthful.

$0 < x < 10$

# Predicates:

Propositions which depend on a variable:

Ex: $P(x) : x \geq 0$

## Negating predicates:

$\neg P(x) : x < 0$

$$\neg(\underbrace{(x>0)}_{p} \wedge \underbrace{(x<10)}_{q}) = \qquad \neg(p \wedge q)$$

$$= \neg p \vee \neg q$$

$$\neg(x>0) \vee \neg(x<10)$$

$$= (x \leq 0) \vee (x \geq 10)$$

# Quantifiers

$\mathbb{R}$ = real
$\mathbb{Z}$ = integers
$\mathbb{N}$ = natural #s

$\forall x\, P(x)$ : For all $x$ (in universe), $P(x)$ is true

Universal

Ex: Let $P(x) =$ "$x+1 > x$"
$Q(x) =$ "$x < 2$"

Give truth values for:

← real numbers

$\forall x \in \mathbb{R}, P(x)$ : For all real #s $x$, $x+1 > x$. T

$\forall x \in \mathbb{R}, Q(x)$ : For all $x$, $x < 2$. F

Existential

## Quantifiers (cont.):

$\exists x\, P(x)$: There exists an $x$ (in universe) such that $P(x)$ is true.

Ex: Let $P(x) =$ "$x+1=x$"
$Q(x) =$ "$x < 2$"

Give truth values for:

$\exists x \in \mathbb{R},\ P(x)$: F

"There is an $x$ with $x+1=x$."

$\exists x \in \mathbb{R},\ Q(x)$: T  Ex: 1

There is an $x$ with $x<2$.

These can get more complicated:

$$\exists x \left( P(x) \wedge Q(x) \right) \vee \forall x R(x)$$

Which quantifier holds where?

$$\exists x \left( P(x) \wedge Q(x) \right) \vee \forall y R(y)$$

## Negations:

How should we negate quantifiers?

Consider: $P(x) =$ "x has taken college algebra."

$\forall x P(x)$: Everyone has taken college algebra.

What is $\neg(\forall x P(x))$? Someone has not taken college algebra.

$$\neg(\forall x P(x)) = \exists x \neg P(x)$$

What about $\exists x\, P(x)$:

Some one has taken college algebra.

$\neg(\exists x\, P(x))$: No one has taken college algebra.

## Negation rules:

$$\neg(\exists x P(x)) = \forall x \neg P(x)$$

$$\neg(\forall x P(x)) = \exists x \neg P(x)$$

And they "stack":

$$\neg(\forall x \forall y P(x,y)) = \exists x \exists y P(x,y)$$

$$\neg(\forall x (P(x) \vee Q(x))) = \exists x \neg(P(x) \vee Q(x))$$

$$= \exists x(\neg P(x) \wedge \neg Q(x))$$

Nested quantifiers:

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, (x+y=0)$:

translate: For any $x$, there is a $y$, with $x+y=0$.

T

$\exists y \, \forall x \, (x+y=0)$: F

There is a $y$ s.t. for all $x$,

$x+y=0$

Another:

Suppose are universe is $\mathbb{R}$. T

Translate: $\forall x \, \forall y \Big( \big( (x>0) \wedge (y<0) \big) \rightarrow (xy<0) \Big)$

For all x and all y, if x is positive and y is negative, then xy is negative.

# Negating implications

What is $\neg(p \to q)$?

| $p$ | $q$ | $p \to q$ | $\neg(p \to q)$ | $\neg q$ | $p \wedge \neg q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | F |
| T | F | F | T | T | T |
| F | T | T | F | F | F |
| F | F | T | F | T | F |

equivalent

So:

What is $\neg\left(\forall x^{\in\mathbb{R}}\left(P(x) \to Q(x)\right)\right)$?

$$\exists x\, \neg\left(P(x) \to Q(x)\right)$$

$$\exists x^{\in\mathbb{R}}\left(P(x) \wedge \neg Q(x)\right)$$

Ex: Write the negation of
$\forall x>0$, if $x^2=1$, then $x^3=1$.

There is an $x>0$ with $x^2=1$ and $x^3\neq 1$.

# Proofs:

A theorem (or lemma, proposition, etc.) is a statement that can be rigourously shown to be true.

Generally something like:

"If $n$ is even, then it is divisible by 2."

"$\forall \varepsilon > 0 \ \exists \delta > 0$ such that if $|f(x) - f(y)| < \varepsilon$, then $|x - y| < \delta$."

The sequence of statements giving that arguement is called a proof.

# Direct proofs

Think about $p \to q$.

When is it false?

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | |
| T | F | |
| F | T | |
| F | F | |

Ex: If $n$ is an odd integer, then $n^2$ is even.

true or false?

Ex: If $n$ is an even integer, then $n^2$ is even.

Proof: (Assume $p$ is true, & show $q$ can't be false.)