# Math 135 - Even more induction!

Note Title 10/12/2012

## Announcements

- Office hours today: 3-4 / (not 1-2)

- HW is up - due next Wed. at the start of class

- Come collect worksheets

# Worksheet problem

Prove that $1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ when $n$ is a positive integer.

pf: Base case: $n=1$

$LHS = 1\cdot 2 = 2$ $\qquad$ RHS: $\frac{1(1+1)(1+2)}{3} = 2$ ✓

IH: Assume $\underbrace{1\cdot 2 + 2\cdot 3 + \cdots + (n-1)\cdot n}_{= \sum_{i=1}^{n-1} i(i+1)} = \underbrace{\frac{(n-1)(n)(n+1)}{3}}$

IS: $1\cdot 2 + 2\cdot 3 + \cdots + n\cdot(n+1) = \sum_{i=1}^{n} i(i+1)$

$$= \underbrace{(1\cdot 2 + 2\cdot 3 + \cdots + (n-1)(n))}_{\sum_{i=1}^{n-1} i\cdot(i+1)} + n(n+1)$$

Apply IH: $= \frac{(n-1)(n)(n+1)}{3} + n(n+1)$

$$= \frac{(n-1)(n)(n+1) + 3n(n+1)}{3} = \frac{n(n+1)(n-1+3)}{3}$$

$$= \frac{n(n+1)(n+2)}{3}$$

$\blacksquare$

Show that for all $n \geq 0$, $10^n - 1$ is divisible by 9.

**Base case:** $n = 0$ $\quad 10^0 - 1 = 1 - 1 = 0$ ✓

**IH:** Assume $\underbrace{(10^{n-1} - 1)}_{=9k}$ is div. by 9

$$10^{n-1} = 9k + 1$$
$$\Downarrow$$
$$10^{n-1} - \frac{1}{10} = 9k + \frac{9}{10}$$

**IS:** $10^n - 1 = 10 \cdot \left(10^{n-1} - \frac{1}{10}\right)$

$$= 10\left(9k + \frac{9}{10}\right)$$
$$= 90k + 9$$
$$= 9(10k+1)$$

So div. by 9

[let $x = 10^{n-1}$. we have $10x = x + 9x$]

IH: Assume $10^{n-1} - 1$ div by 9

IS: $10^n - 1 = 10 \cdot 10^{n-1} - 1$

$= (1 \cdot 10^{n-1} + 9 \cdot 10^{n-1}) - 1$

$= 9 \cdot 10^{n-1} + \underbrace{10^{n-1} - 1}_{\text{IH} \Rightarrow \text{div by } 9}$

Set Theory proof:

Use induction to show that if $S$ is a finite set of $n$ elements, then $\mathcal{P}(S)$ has $2^n$ sets in it.

proof: induction on size of $S$, $n$

Base case: $n = 0$, so $S = \emptyset$

$\mathcal{P}(\emptyset) = \{\emptyset\}$ $\quad |\mathcal{P}(\emptyset)| = 1 = 2^0$

$n=1$: $S = \{x\}$ $\quad |S| = 1$

$\mathcal{P}(S) = \{\emptyset, \{x\}\}$ $\quad |\mathcal{P}(S)| = 2 = 2^1$

IH: If $S'$ has $n-1$ elements, $P(S')$ has $2^{n-1}$ elements.

IS: Consider $S$ with $n$ elements, Know there is some element $x$ in $S$.

Any subset of $S$ either has $x$ in it or it doesn't.

Consider $S - \{x\} \Rightarrow$ has $2^{n-1}$ subsets by IH.

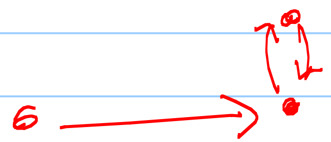
For any one of these, could add $x$ back in & still have subset of $S$.

$\Rightarrow 2^{n-1} + 2^{n-1} = 2^n$ $\blacksquare$

Suppose $n$ friends have a water balloon fight. Each moves to a location (so all distances are unique), + then throws their balloon at the closest person.

<u>Claim:</u> If $n$ is odd, then at least one person stays dry.

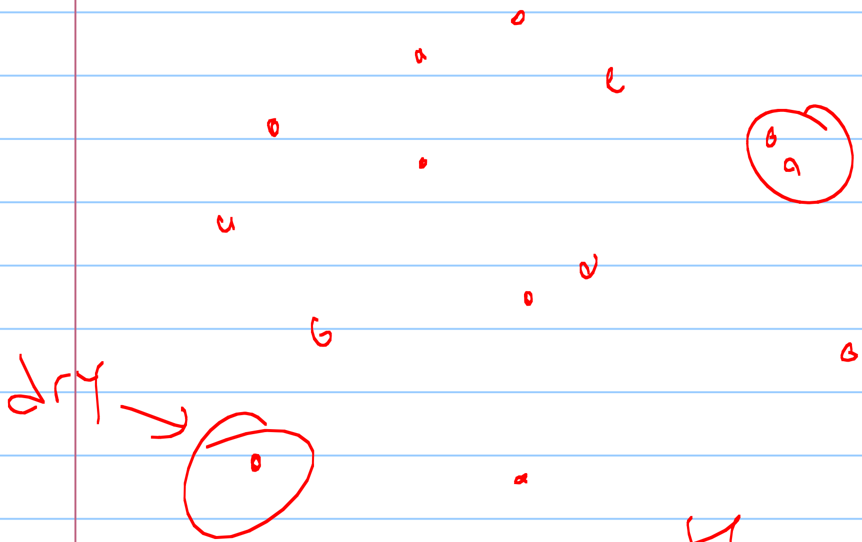
<u>Base case:</u> $n=3$



Closest pair throw at each other, so 3rd guy stays dry.

Assume: $2n-1$ odd people, someone stays dry

IS: Consider $2n+1$ people




Consider (& remove) closest pair
Know someone would stay dry without them.
When readding, people could throw at closest pair
But closest pair always throw at each other.

# The Gossip Problem

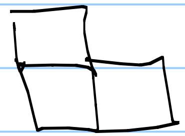
- There are $n$ people, each of whom knows 1 secret.
- Every time 2 people call each other, they tell each other all the secrets they know.

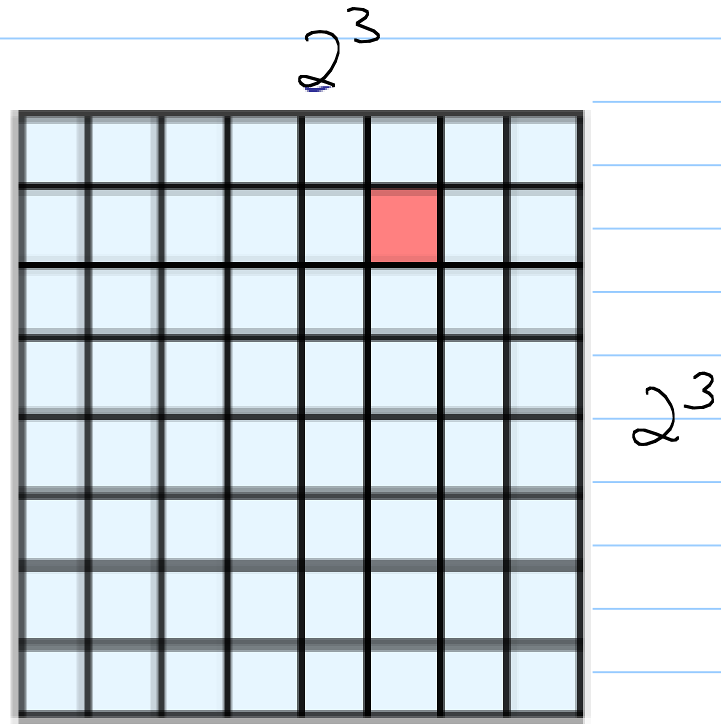
How many phone calls are needed for everyone to know all the secrets?

Claim: If $n \geq 4$, then $2n-4$ suffice.

Let $n$ be a positive integer.

Show that any $2^n \times 2^n$ chess board with 1 tile removed can be tiled with L-shaped pieces:



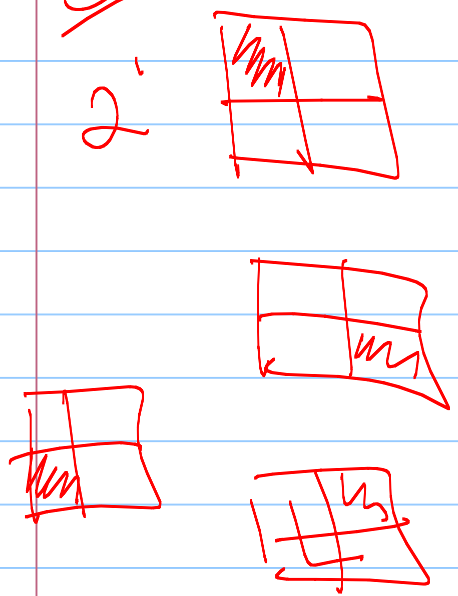


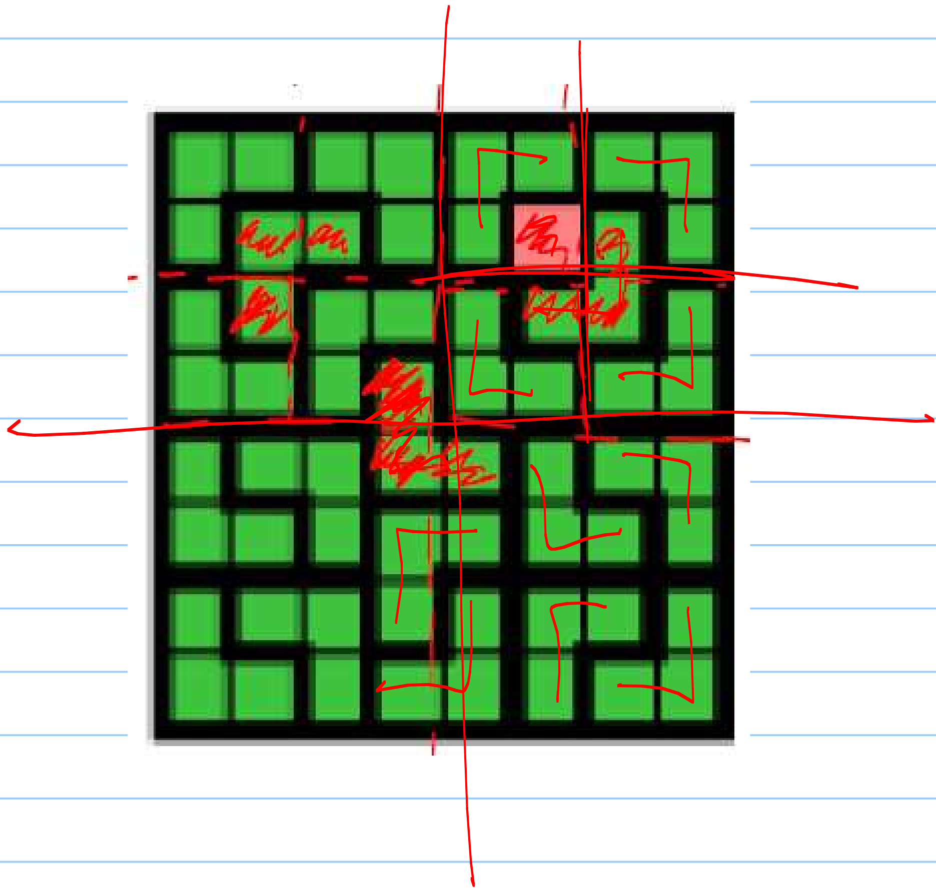
# Example:

Base case

$2^1$

$2^1$

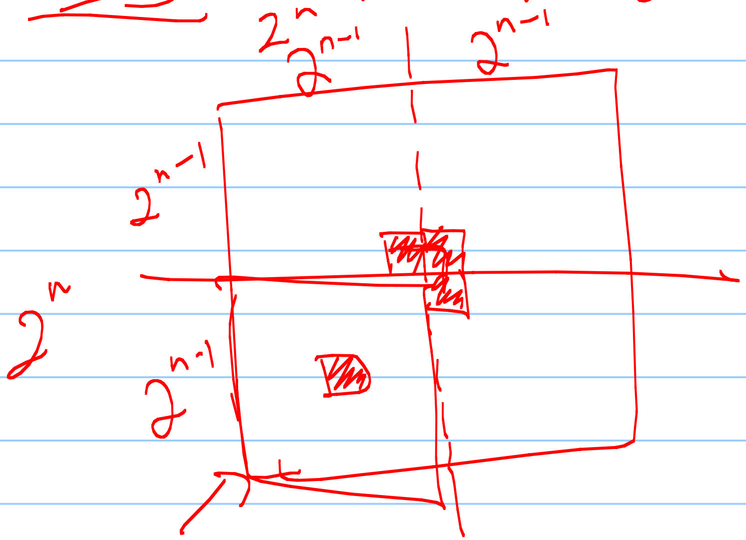




Proof: induction on $n$

IH: Can tile $2^{n-1} \times 2^{n-1}$ w/ a square removed.

IS: Consider $2^n \times 2^n$ board




Cut into 4 $2^{n-1} \times 2^{n-1}$ boards. One already has a square out, so tile it by IH.

Other 3 form an L. Remove a square from corner of each.

Now by IH, tile each of them.