Math 135 - Planar Graphs

Note Title 12/5/2012

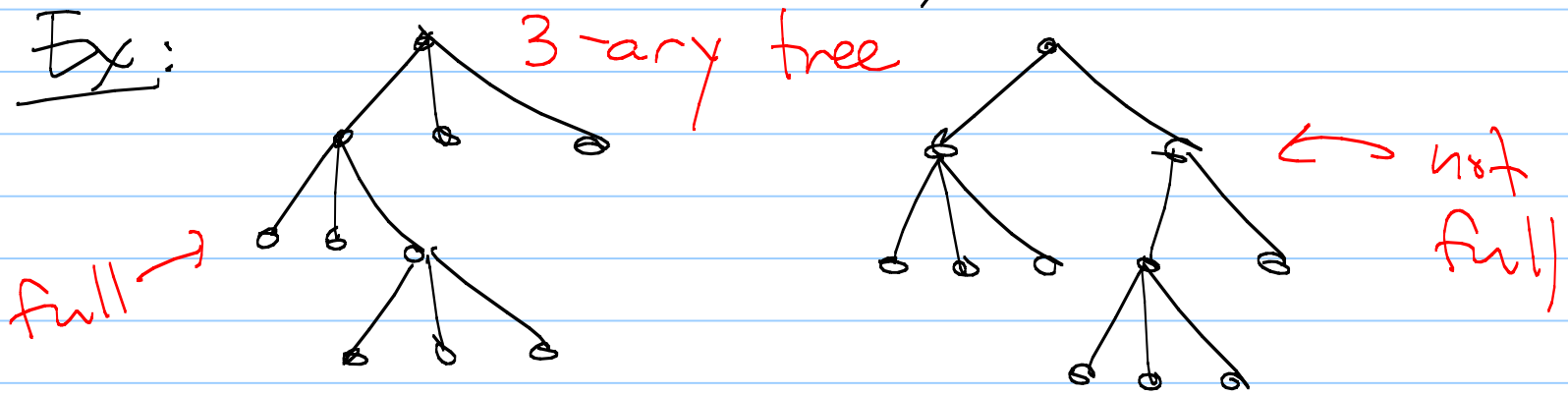
Announcements

M-ary trees

A rooted tree is an m-ary tree if every internal vertex has no more than m children.

An m-ary tree is full if every vertex has exactly m children.

Ex:



Thm: A full m-ary tree with i internal vertices has $n = mi + 1$ vertices in total.

pf: each internal vertex has m children
$\Rightarrow m \cdot i$

Root has no parent, but every other node does

Total: $m \cdot i + 1$

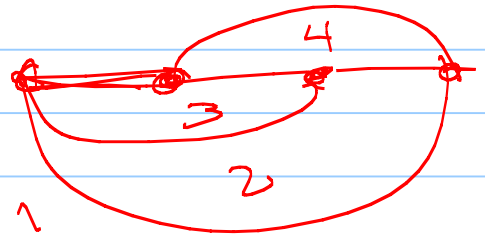
Cor: full binary tree has $2 \cdot i + 1$ vertices

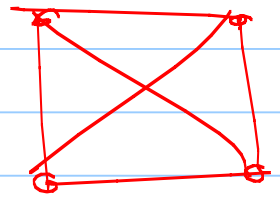
Drawing graphs

A graph is called planar if it can be drawn with no edge crossings.

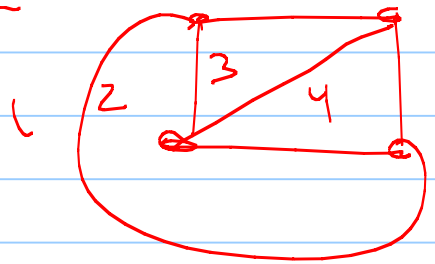
Note: there are many ways to draw a graph. We only need one with no crossings for it to be planar!

Example: K_4



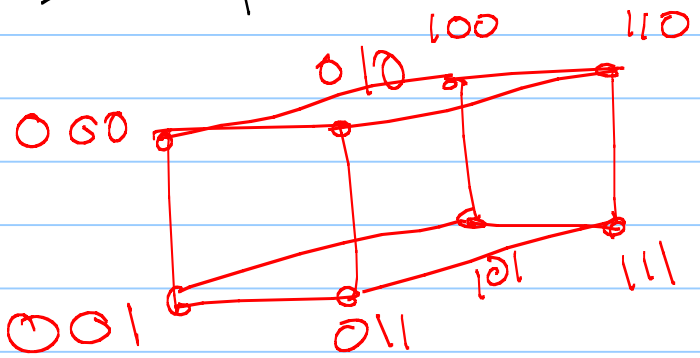


or

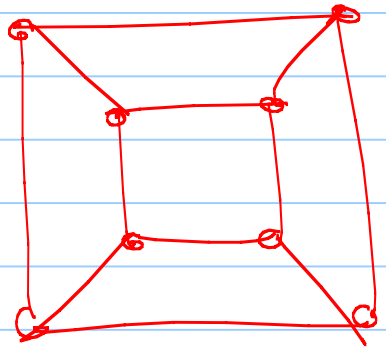


yes

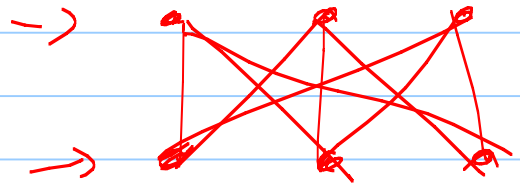
Example: Q_3

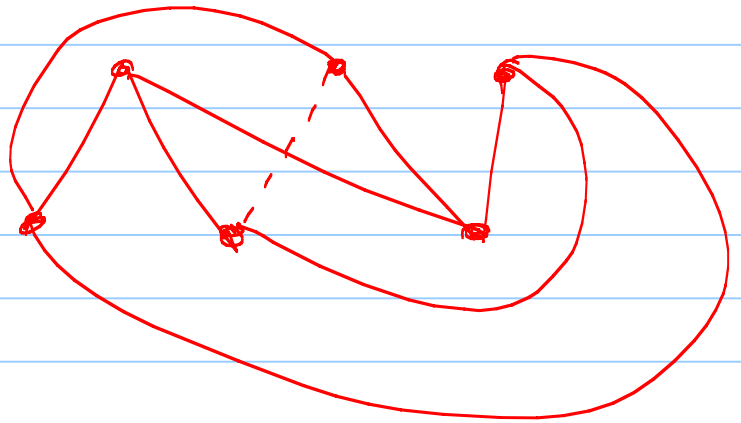


yes



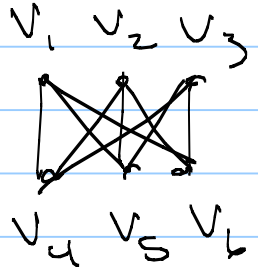
Example: $K_{3,3}$



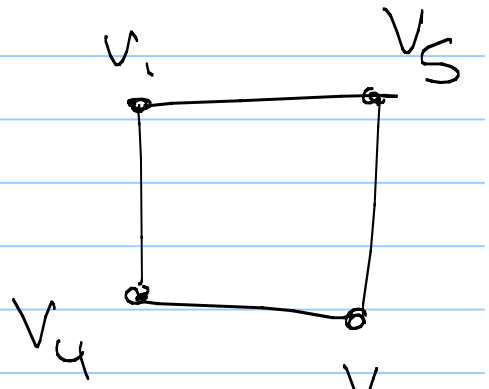


Lemma: Impossible to draw $K_{3,3}$ in the plane.

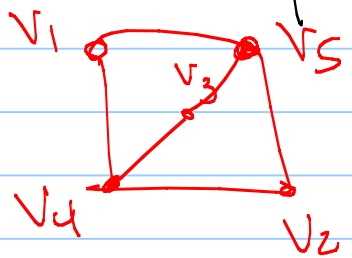
Pf:

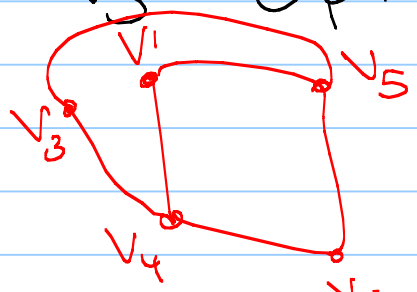


Draw cycle: forms closed curve →



Now place v_3. Options?





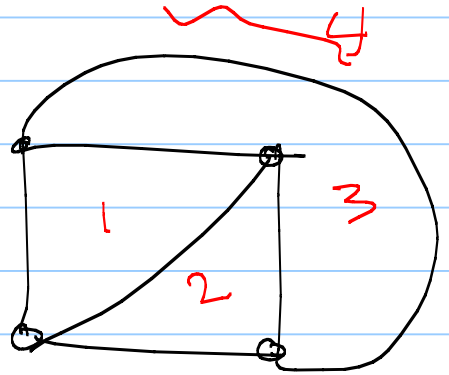
v_6 needs a spot

Any option results in a crossing.

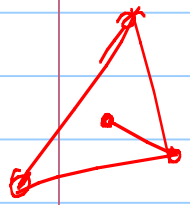
Euler's formula

Since a drawing divided the plane into regions, we can talk about the number of such regions.

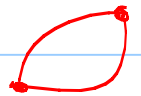
Ex: K_4



4 regions



each region is bounded by ≥ 3 edges



Fact: Any planar drawing of a graph results in the same number of regions.

Euler's Formula: In a planar graph with v vertices and e edges, have

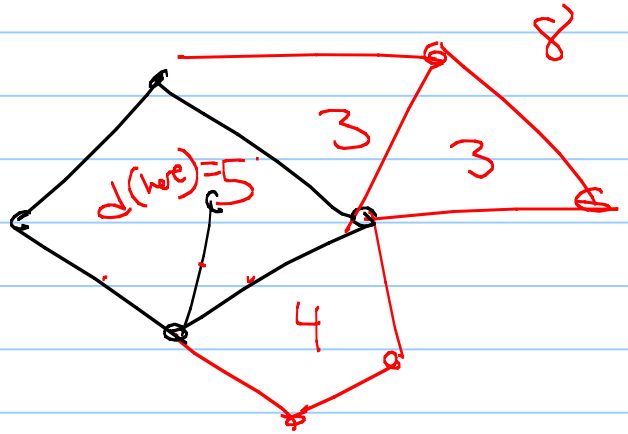
$$\#\text{regions} = e - v + 2$$

(pf in book)

Useful Corollary (see hw!)

In a connected planar graph, $e \leq 3v - 6$.

pf: Consider region:



$$3r \leq \sum_{r \in \text{regions}} d(r) = 2e$$

$d(r) \geq 3$

$\Rightarrow r \leq \frac{2}{3} \cdot e$

& $r = e - v + 2 \Rightarrow e - v + 2 \leq \frac{2}{3} \cdot e$

$\frac{1}{3} \cdot e \leq v - 2 \Rightarrow e \leq 3v - 6$

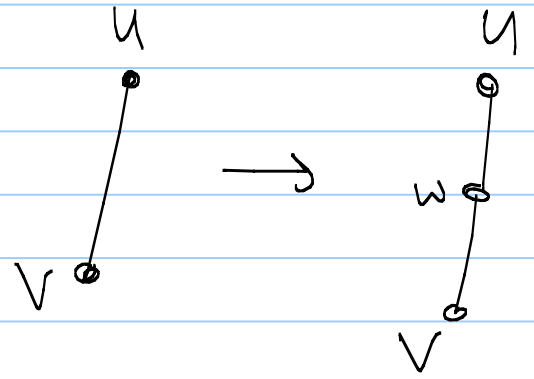
Kuratowski's Thm

↙ on worksheet

Know that $K_{3,3}$ & K_5 are not planar.

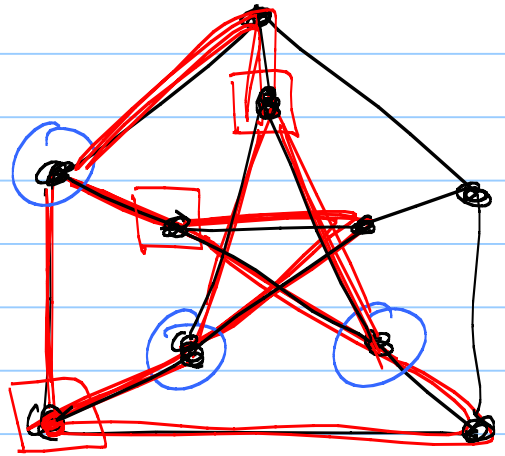
Surprisingly, these are in some sense, the most important planar graphs.

Dfn: Edge subdivision:



Thm: A graph is not planar if and only if it contains a subdivision of $K_{3,3}$ or K_5.

Ex:



Peterson graph