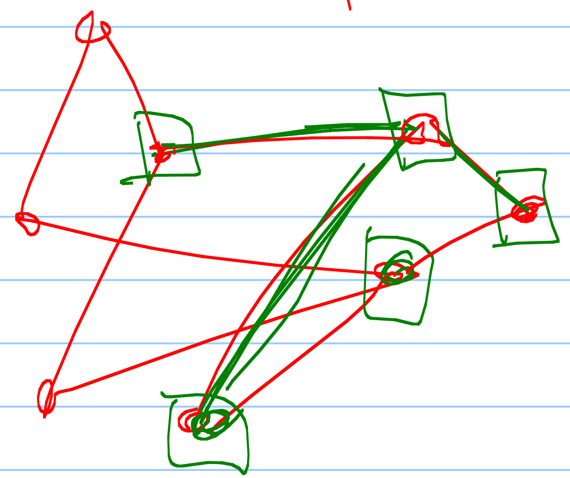
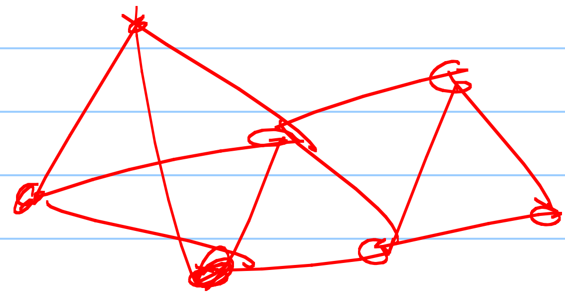
More on graphs

Note Title 11/30/2012

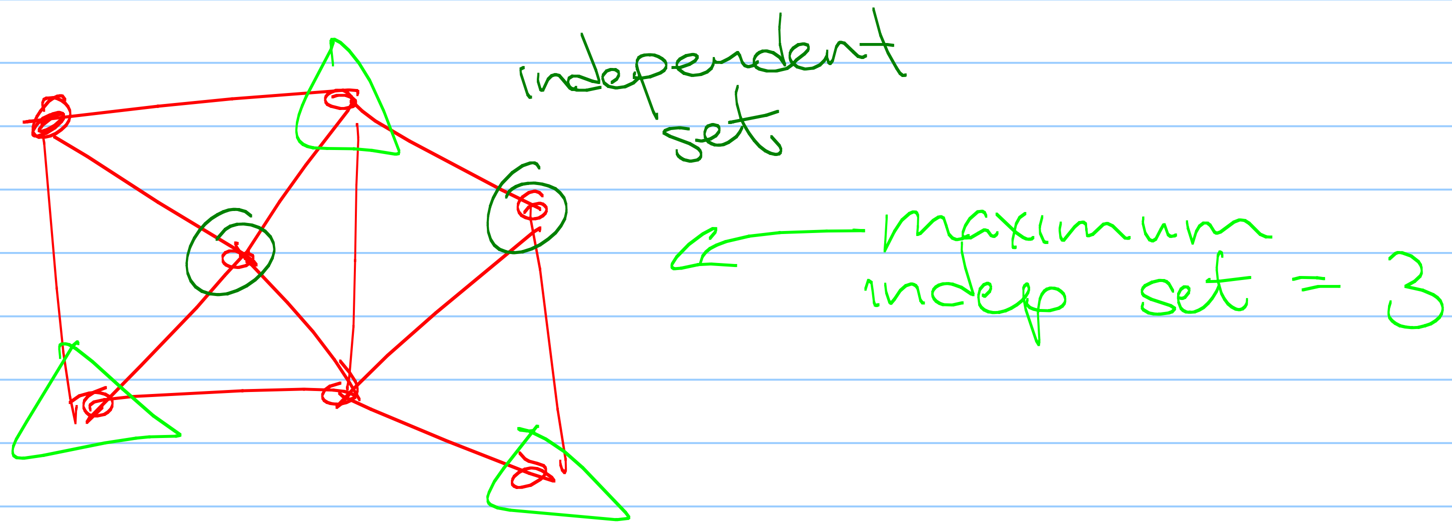
Dfn: Subgraph of $G = (V, E)$ is subset $(V', E') = G'$ with $V' \subseteq V$ and $E' \subseteq E$ s.t. any edge in E' has endpoints in V'.



A clique is a subgraph ~~which~~ forms a complete graph.



An independent set is a subset of vertices with no edges between them.

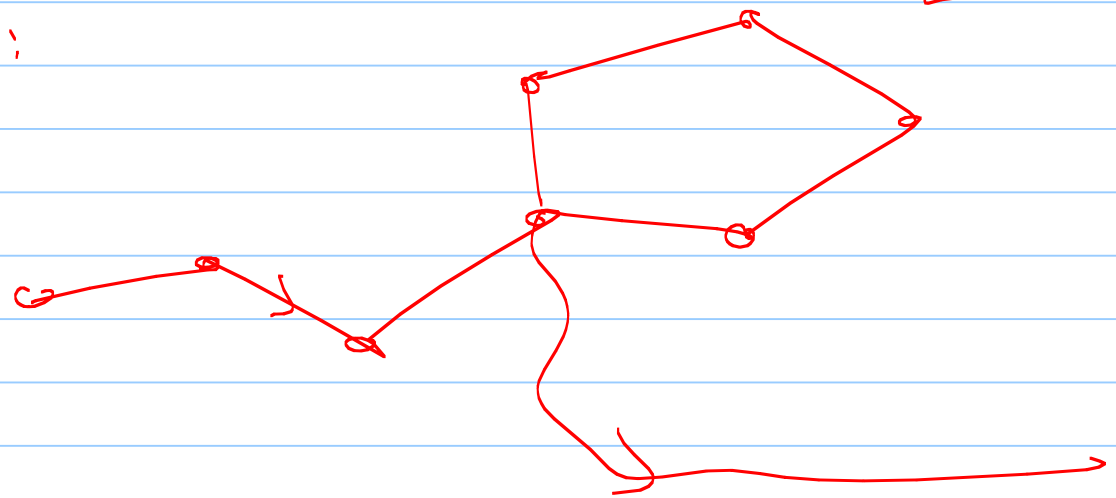


repeat edges + vertices

Lemma: Every u-v walk contains a u-v-path.

$W = v_1 \to v_2 \to v_3 \to v_4 \to \underbrace{v_i \to \cdots \to v_i}_{v_i} \to \cdots$

sketch:



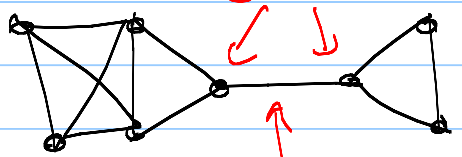
[**Cor:** In a connected graph, every pair of vertices has a simple path between them.]

endpoints stay

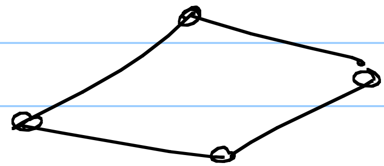
Dfn: A cut edge in a graph is an edge whose deletion increases the number of components.

A cut vertex is a vertex whose deletion increases # of components.

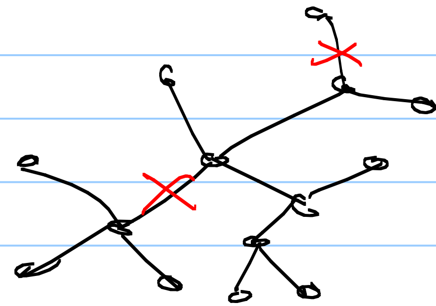
cut vertex



cut edge



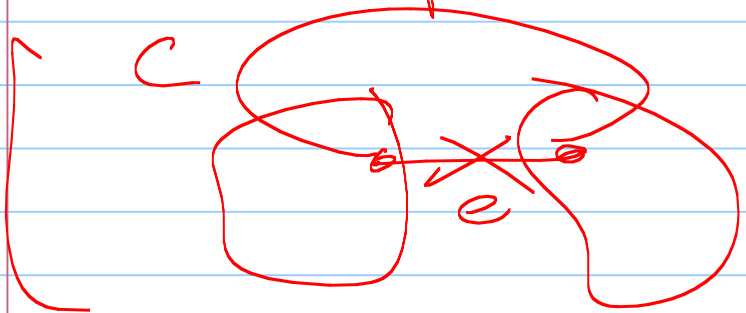
cycle has no cut edge or vertex



all are cut edge

Thm: An edge is a cut edge $\iff$ it does not belong to any cycle.

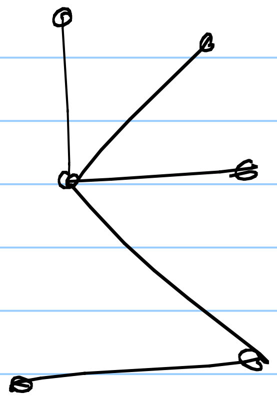
$\Rightarrow$: Say e is a cut edge.
Suppose e belongs to a cycle C.



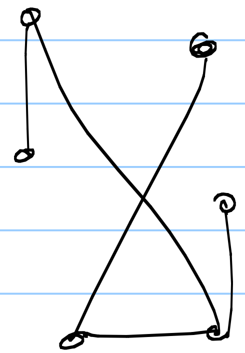
G − e is disconnected.
Some pair u, v used e on their connecting path.
Replace e with C − e
+ u & v are still connected
$\Rightarrow$ e was not a cut edge ↯.

Trees

Dfn: A tree is a connected graph with no cycles.

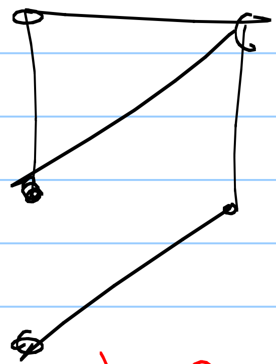


tree

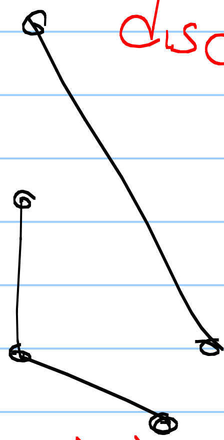


tree

has cycle



not a tree



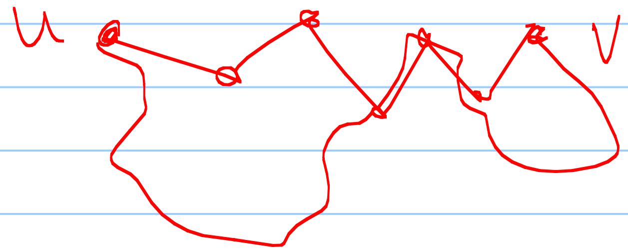
disconnect

No

Thm: An undirected graph is a tree if & only if there is a unique, simple path between any two vertices.

$\Rightarrow$: Suppse G is a tree

Suppose u & v have 2 paths.



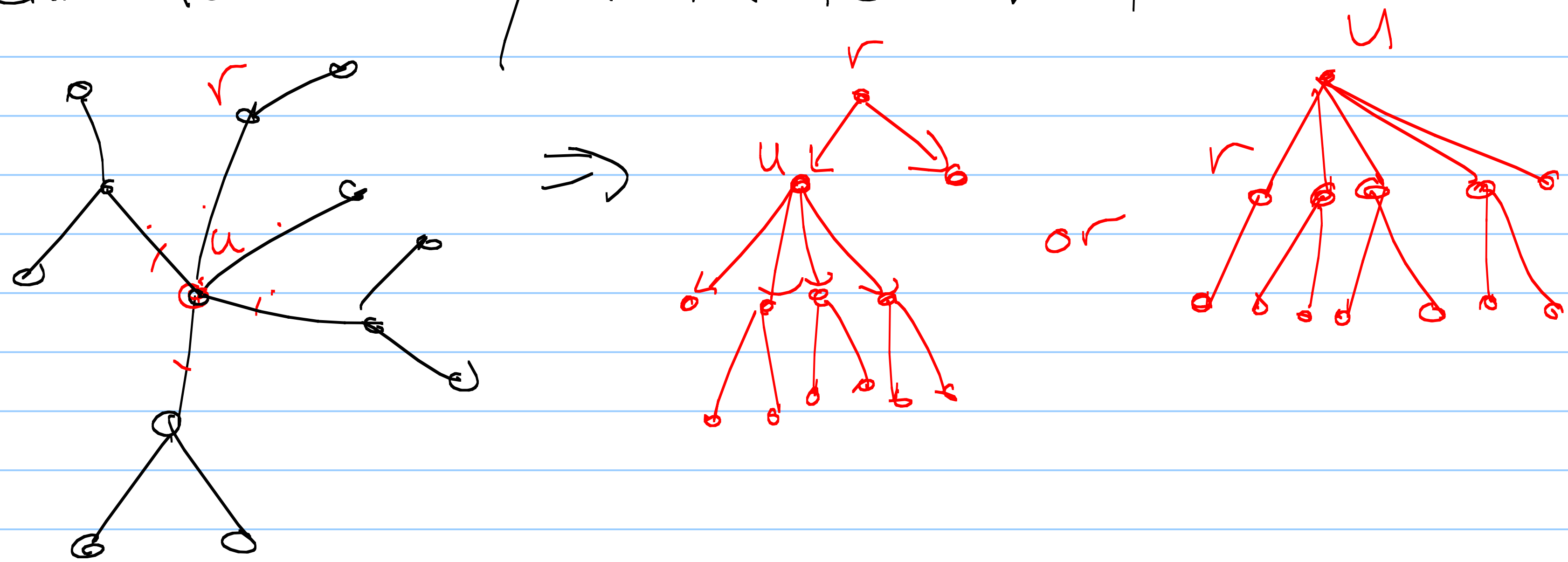
a second path creates at least one cycle.

$\Leftarrow$: leave as ex.

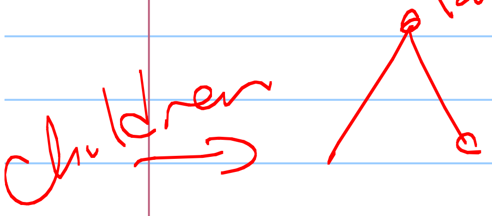
A Graph is a tree <u>iff</u>:

- unique path between any 2 vertices
- connected & has n-1 edges
- n-1 edges & no cycles

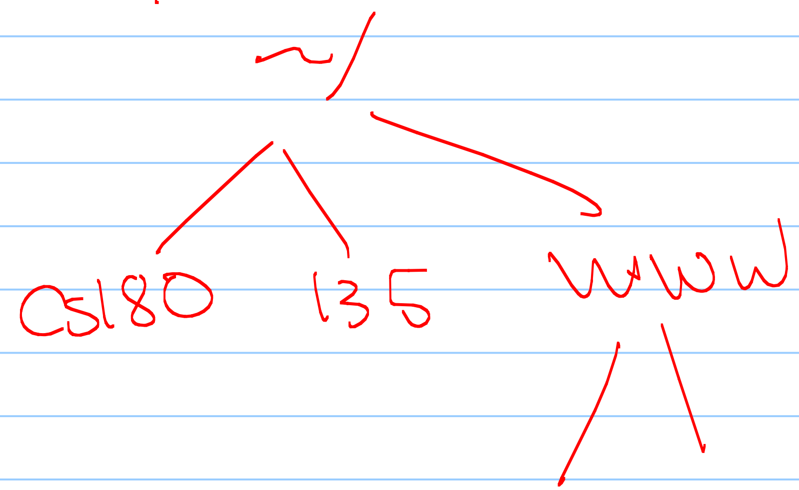
Dfn: We often work with <u>rooted trees</u>, where one vertex has been designated as a root and every edge is directed away from the root.



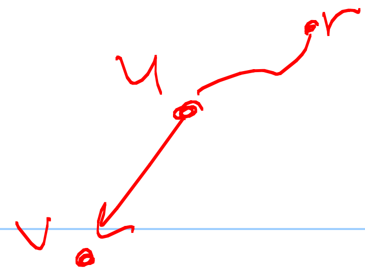
Family tree



file system



Terminology



parent of v : The vertex u such that there is an edge $u \to v$.
(v is called the **child** of u)

Also: **siblings** : share same parent

ancestors

descendants

leaf : has no children

internal vertex : has children

__Lemma__: Every tree has ~~a leaf~~ 2 leaves, (& removing leaf leaves a smaller tree)

__pf__:

think about path in tree

extend to __maximal__ path

endpoints are leaves.

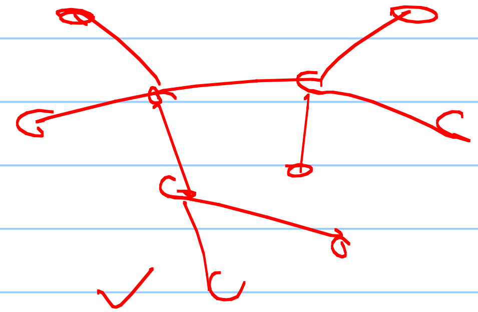
Lemma: Every edge in a tree is a cut edge.

Use prev thm & note that no edge belongs to a cycle in a tree.

Lemma: Adding an edge to a tree forms exactly one cycle.

$$|E| = m = O(n)$$

Thm: A tree with n vertices has $n-1$ edges.



pf: induction on n:

base case: $n=1$ ✓

0 edges

IH: a tree with $k < n$ vertices has $k-1$ edges

IS: Consider tree w/ n vertices. Remove a leaf to get T' with $n-1$ vertices. By IH, T' has $n-2$ edges. T had one more vertex + 1 more edge. So T had $n-1$ edges.

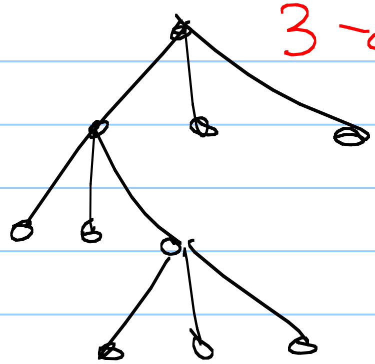
M-ary trees

A rooted tree is an m-ary tree if every internal vertex has no more than m children.

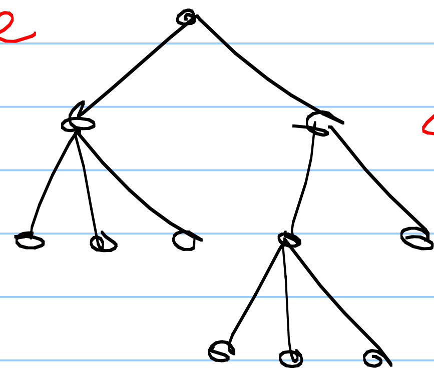
An m-ary tree is **full** if every vertex has exactly m children.

Ex:

3-ary tree



full



not full

Thm: A full m-ary tree with i internal vertices has $n = mi + 1$ vertices in total.