Graphs

Note Title 11/26/2012

Announcements

- HW due Friday
- Last HW due last day of class

Graphs – Ch. 10

Motivation: Model relationships or connections

- Cities + roads
- Internet connectivity (routers, computers, etc...)
- Webpage Links
- Social Networks
- Biological Networks

⋮

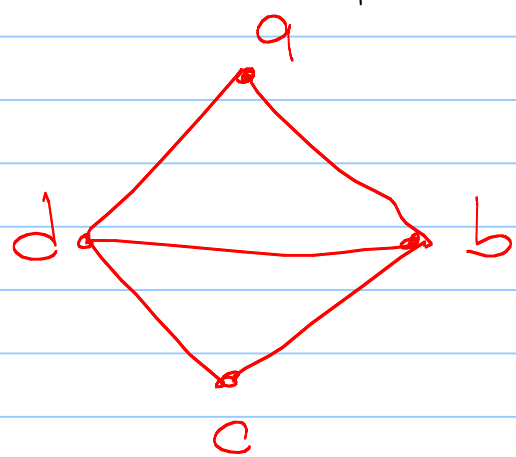
Dfn: A graph $G=(V,E)$ is a pair of sets:

- V is a set of vertices
- E is a set of edges

Each edge is a set of 2 vertices, called its endpoints.

Ex: $V=\{a,b,c,d\}$

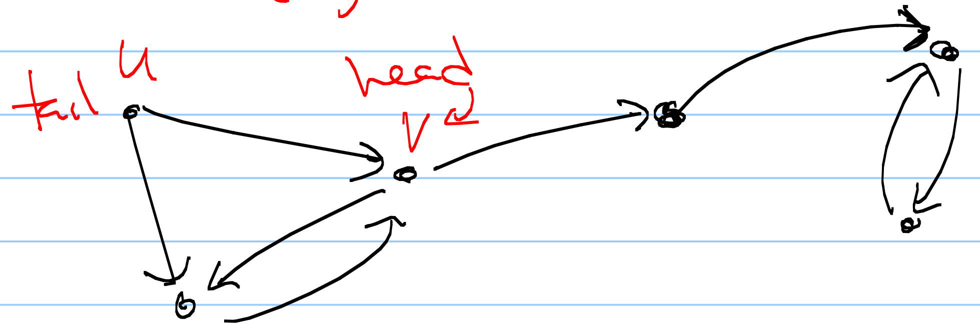
$E=\{\{a,b\},\{b,c\},\{c,d\},\{a,d\},\{b,d\}\}$



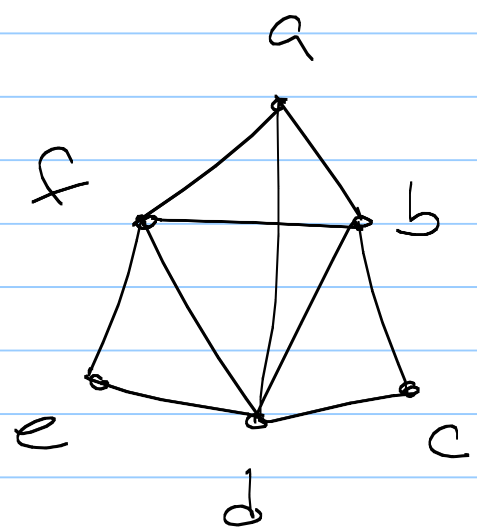
In a directed graph, each edge is an ordered pair - not just a set.

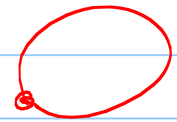
Ex:

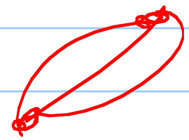
$(u, v) \in E$



We say an edge is <u>incident</u> to its endpoints, and two vertices are <u>adjacent</u> if there is an edge between them.



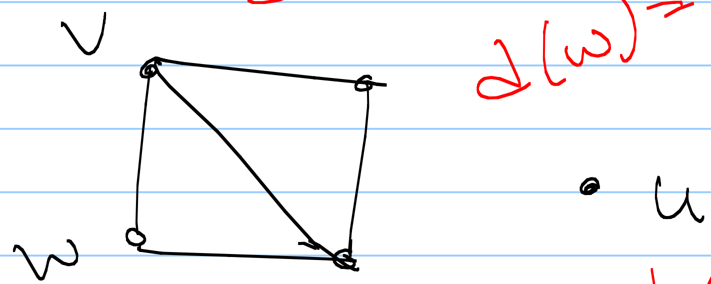
We can have loops:  $\{v, v\}$

or multiple edges: 

A graph is called simple if it has no loops or multiple edges.

We'll (usually) deal with simple, undirected graphs here.

Dfn: The degree of a vertex, $d(v)$, is the number of incident edges.

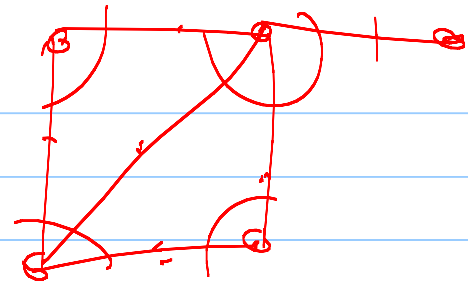


$d(v) = 3$

$d(w) = 2$

$d(u) = 0$

degree of a vertex is between 0 and $n-1$



Thm: $$\sum_{v \in V} d(v) = 2|E|$$

pf: combinatorial proof

RHS: Counting each edge twice

LHS: Each edge is a set of 2 vertices so it counts +1 for two $d(v)$'s, So over all vertices, each edge is counted twice here also.

$3+2+4+2+1$
$= 12$

$2|E| = 2 \cdot 6$

Thm: In a simple, undirected graph, the number of nodes with odd degree is even.

pf: Pf by contradiction:

Sppse an odd # of odd degree nodes.

Then sum of degrees will be odd.

but $\sum d(v) = 2|E| \Rightarrow$ must be even.

$\lightning$

$\blacksquare$

Dfns:

A **walk** is a list of vertices $v_1, v_2, \dots, v_k$ where each $\{v_i, v_{i+1}\} \in E$.

A **path** is a walk with no repeated vertices or edges.

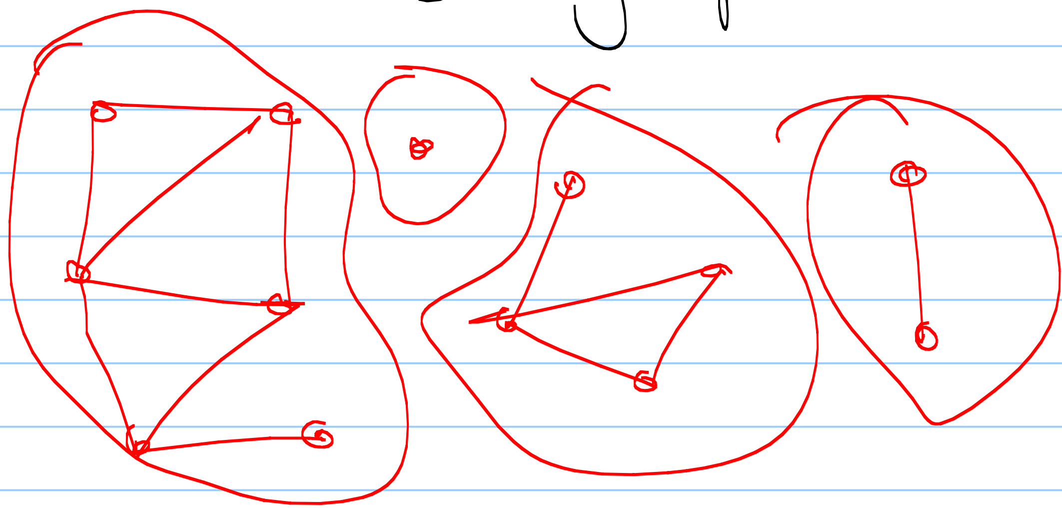
A **cycle** is a path except at start vertex = end vertex.

A **circuit** is a walk that ends where it begins.
(can repeat edges & vertices)

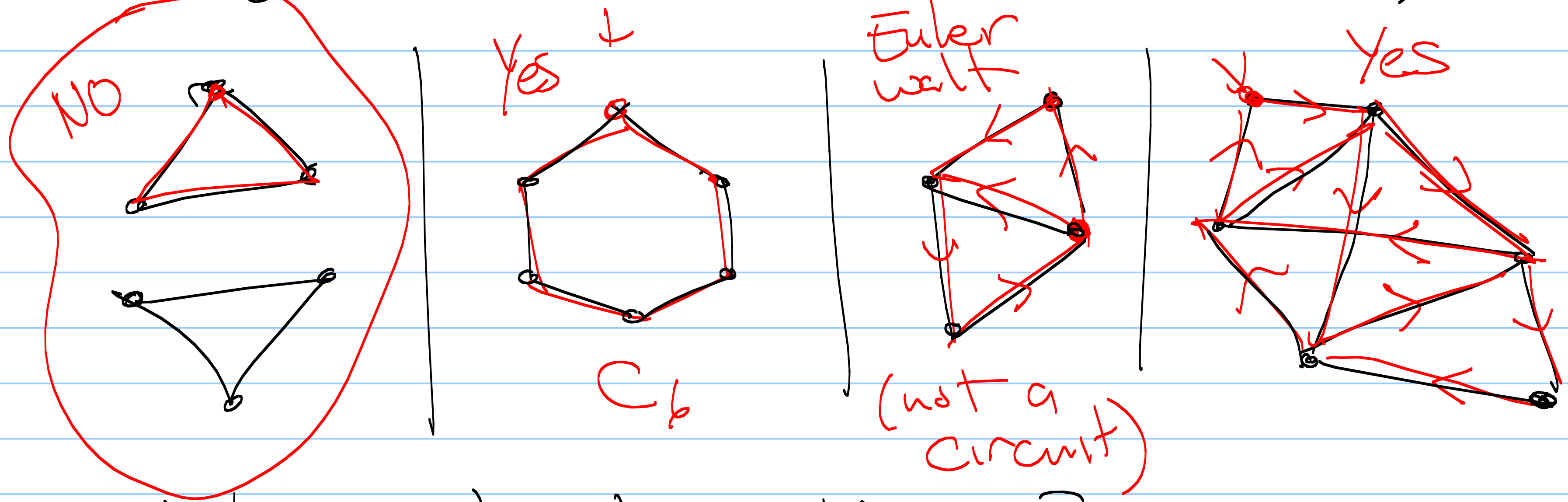
Dfn: A graph is connected if for every pair of vertices u & v, there is a u-v walk in G.

The components of G are maximally connected subgraphs.

4 components



Dfn: An Eulerian circuit is a circuit which uses every edge exactly once

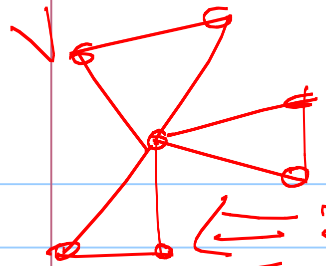


What graphs have these?

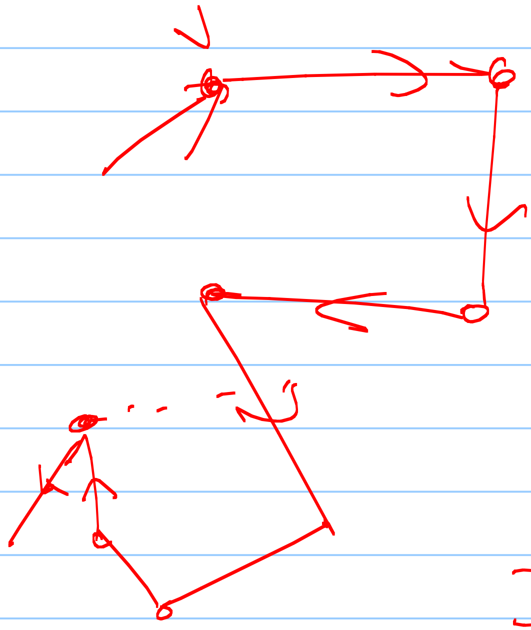
Thm: A graph has an Eulerian circuit $\iff$ G is connected and every vertex has even degree.

pf: $\Rightarrow$: Assume G has euler circuit. Circuit gives a walk between any of vertices, so all vertices must be connected.

Every time circuit visits a vertex, "adds" +2 to degree, so total degree must be even.



$\Leftarrow$: G connected + even degree.
Pick a vertex $v \in V$.
Follow any edge $\{v, u\}$

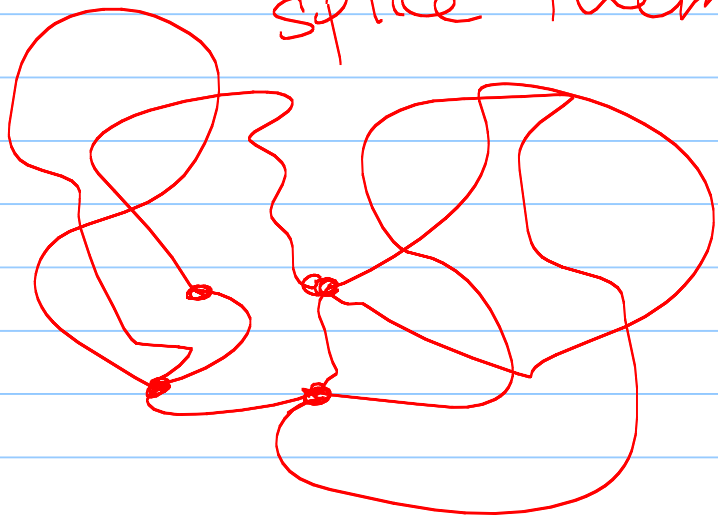


Each time you visit a vertex $\neq v$, must be able to leave it, since you "use" 2 edges with each visit.

Can get "stuck" at v.
If has visited all edges, done.
If not, "erase" the edges visited & repeat.

End result is a collection of circuits which cover the graph.

Since they cover the graph & graph is connected, they intersect, so we can splice them together.



This gives an Euler tour.