Math 135 - Sequences

Note Title 9/19/2012

Announcements

- HW1 back today
- HW3 due Friday
- Midterm in 2 weeks

Ex: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

looks true, so prove it.

① $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

② $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

①: take $x \in A \cup (B \cap C)$

by dfn of $\cup$, $x \in A$ or $x \in B \cap C$

by dfn of $\cap$, $(x \in A) \vee (x \in B \wedge x \in C)$

①: take $x \in A \cup (B \cap C)$

by dfn of $\cup$, $x \in A$ or $x \in B \cap C$

by dfn of $\cap$, $(x \in A) \vee (x \in B \wedge x \in C)$

using logic rule, $p \vee (q \wedge r) = (p \wedge q) \vee (p \wedge r)$

$(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$

by dfn,

$x \in A \cap B \vee x \in A \cap C$

& by dfn, $x \in (A \cap B) \cup (A \cap C)$

$A \times C$

$\Rightarrow \quad x = (a, c)$ where $a \in A$, $c \in C$

if $x \in A$ then $x \in B$

Assume $x \in A$

⋮

$\Rightarrow x \in B$

direct proof

<u>OR</u>

Assume $x \notin B$

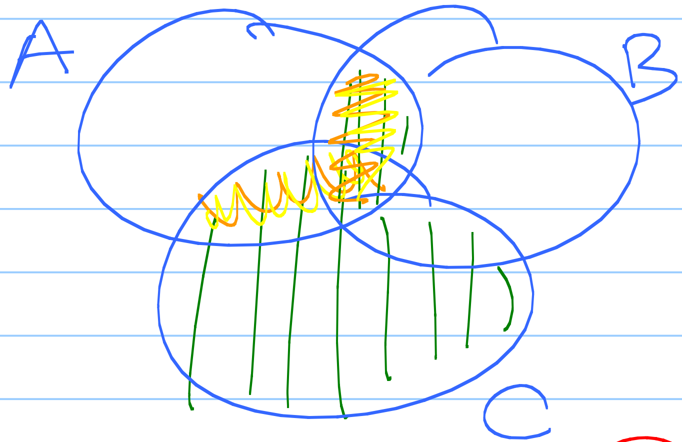
⋮

$x \notin A$

contrapositive

Ex: $\underbrace{A}_{\{1,4\}} \cap \underbrace{(B \cup C)}_{\{1,2,3,5\}} = \{1\}$

$\underbrace{(A \cap B)}_{\{1\}} \cup \underbrace{C}_{\{1,2,3\}} = \{1,2,3\}$



$A = \{1, 4\}$
$B = \{1, 5\}$
$C = \{1, 2, 3\}$

} Disproves the claim.

Sequences & Summations (2.4)

Dfn: A sequence is a function from a ~~subset~~ of $\mathbb{Z}$ (usually $\mathbb{N}$) to a set S.

a_n = image of n under function
= n^{th} term of sequence

Ex: $a(n) = a_n = \frac{1}{n}$, or $\{\frac{1}{n}\}_{n \in \mathbb{N}}$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

Types:

A geometric progression is a sequence of the form:

$$a, \quad ar, \quad ar^2, \ldots, ar^n, \ldots$$

Ex:

$b_n = (-1)^n = \overset{a}{1} \cdot (\overset{r}{-1})^n$: $1, -1, 1, -1, \ldots$

$c_n = 2 \cdot 5^n$: $2, 10, 50, 250, \ldots$

$d_n = 6 \cdot \left(\frac{1}{3}\right)^n$: $6, 2, \frac{2}{3}, \frac{2}{9}, \ldots$

Types:

An arithmetic progression is a sequence of the form

$$a, a+d, a+2d, \dots, a+nd, \dots$$

Ex:

$\mathbb{N}$: 0, 1, 2, 3, 4, ...

$s_n = -1 + 4n$: −1, 3, 7, 11, ...

$t_n = 7 - 3n$

Recurrences

A **recurrence relation** for $\{a_n\}$ is an equation that expresses a_n in terms of $a_{n-1}, \ldots, a_1$.

Ex: $a_n = a_{n-1} + 3, \quad a_0 = 2$

$2, 5, 8, 11, 14, \ldots$

$\{2 + 3d\}_{d \in \mathbb{N}}$ ⟵ closed form

Fibonacci #s

Ex: $f_0 = 0,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Closed form: not in terms of $a_{n-1}, \ldots, a_1, a_0$

not always obvious

Why we care:

recurrences are connected to divide & conquer algorithms

Summations

We often consider summing sequences:

$$\sum_{i=0}^{n} a\cdot r^i = a + ar + \dots + ar^n$$

Thm: $$\sum_{i=0}^{n} a\cdot r^i = \begin{cases} \dfrac{a\cdot r^{n+1} - a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

$(111111)_2$ $$\sum_{i=0}^{n} 1\cdot 2^i = \frac{2^{n+1}-1}{1} = 2^{n+1}-1$$

Double Summations

$$\sum_{i=1}^{4}\left[\sum_{j=1}^{3} i\cdot j\right] = \sum_{i=1}^{4}\left[i+2i+3i\right]$$

$$= \sum_{i=1}^{4} 6i = 6\left(\sum_{i=1}^{4} i\right)$$

$$= 6(1+2+3+4) = 60$$

Another:

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{i} 1\right) = \sum_{i=1}^{n}\left(\overbrace{1+1+1\cdots+1}^{i \text{ times}}\right) = \sum_{i=1}^{n} i$$

$$= 1+2+3+4+\cdots+(n-2)+(n-1)+n$$

$$= \frac{n(n+1)}{2}$$

Infinite Sets: (ch 2.5)

Dfn: Two sets have the same cardinality $\iff$ there is a bijection from A to B.

Thm: $\mathbb{N}$ & $\mathbb{Z}$ have same cardinality.

$\mathbb{N}$		$\mathbb{Z}$
0	$\mapsto$	0
1	$\mapsto$	1
2	$\mapsto$	2
	$\vdots$	

$(-1)^n \cdot n$

↑ don't hit negatives