Math 135 - Functions (pt 2)

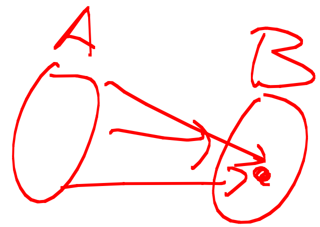
Note Title 9/14/2012

Announcements

- HW 3 is up
- HW 1 - back Wednesday
- This week: finish Ch 2 & start Ch 3 (midterm coming up - will include Ch. 3)

Function (recap)

$f: A \to B$



Functions map elements from 1 set to another.

Dfns:

- domain: 1st set
- codomain (versus range)
  - codomain: 2nd set
  - range: elements in B that are "hit" by f
- onto: $\forall b \in B, \exists a \in A$ s.t. $f(a) = b$
- 1-1: if $f(a) = f(b) \Rightarrow a = b$
- bijection: 1-1 and onto

Dfn: The identity function on a set A, $i_A : A \to A$, is the function $i_A(a) = a$, $\forall a \in A$.

Ex: $i_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$

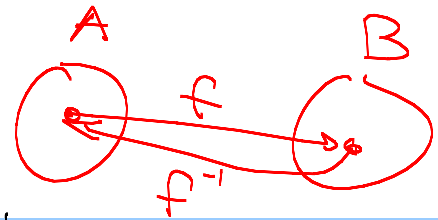
$i_{\mathbb{N}}(x) = x$

$A = \{a, b, c\}$

$i_A(a) = a$

$i_A(b) = b$

$i_A(c) = c$



Dfn: Spps f is a bijection. The inverse of f, written f^{-1}, is the function:

$$f^{-1} : B \longrightarrow A \quad \text{where:}$$

$$f^{-1}(b) = a \iff f(a) = b$$

Ex: What is the inverse of $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x + 1$?

$$y = x + 1$$

$$\Downarrow$$

$$x = y - 1 \quad \leftarrow \text{inverse map}$$

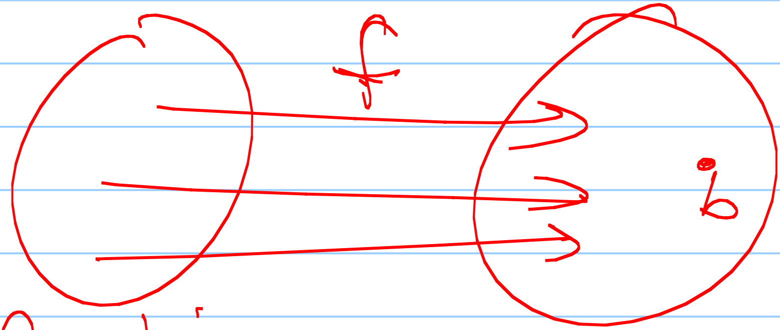
$$f^{-1}(y) = y - 1$$

Note: Only bijective functions have inverses.

Why?

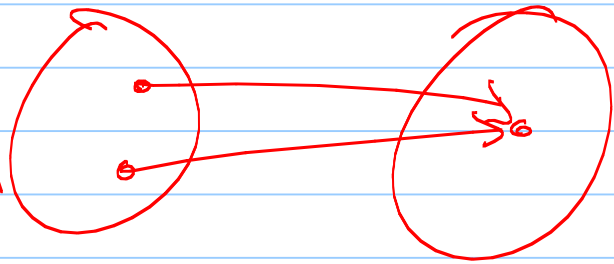
not onto:

some element of B not hit by f, so f^{-1} won't be a function



not 1-1:

again, f^{-1} not a function



Is it 1-1? $f(x) = x^2$

$f(y) = f(x)$

$y^2 = x^2$

$y = \cancel{x}$ NO

Ex: $f: \mathbb{Q} \to \mathbb{Q}$, $f(x) = \frac{x}{2} + 3$

Is it a bijection? (Prove or disprove).

Prove 1-1 and onto:

1-1: if $f(a) = f(b)$ then $a = b$

Suppose $f(x) = f(y)$

So $\frac{x}{2} + 3 = \frac{y}{2} + 3$

subtract 3: $\frac{x}{2} = \frac{y}{2}$

mult by 2: $x = y$ ▫

onto: $\forall \frac{p}{q} \in \mathbb{Q}$, $\exists x$ s.t. $f(x) = \frac{p}{q}$

So $\frac{x}{2} + 3 = p/q$

$\Rightarrow \frac{x}{2} = \frac{p}{q} - 3$

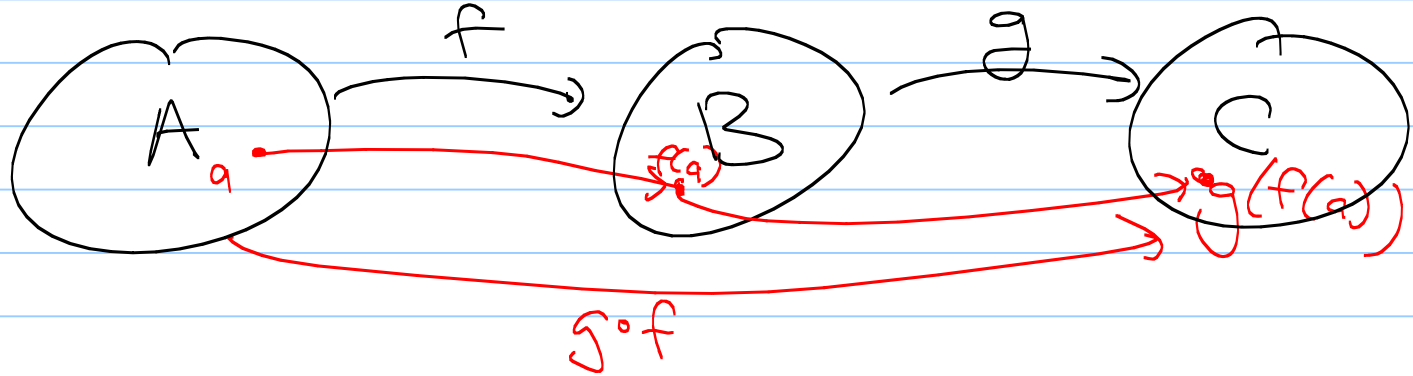
$\Rightarrow x = 2\left(p/q - 3\right) = \frac{2p}{q} - 6 = \frac{2p - 6q}{q}$

Composition of functions

Given $f: A \to B$ and $g: B \to C$, the composition of f ~~and~~ g, written $g \circ f$, is the function

$$(g \circ f)(a) = g(f(a))$$

so $g \circ f : A \to C$



Ex: Let $f: \mathbb{Z} \to \mathbb{Z}$ with $f(x) = 2x+3$
$g: \mathbb{Z} \to \mathbb{Z}$ with $g(x) = 3x+2$

What is $g \circ f$?

$$g \circ f(a) = g(f(a)) = g(2a+3)$$
$$= 3(2a+3)+2$$
$$= 6a+11$$

($\Leftarrow$ is in book)

Thm: Functions $f: A \to B$ and $g: B \to A$ are inverses $\Longrightarrow$

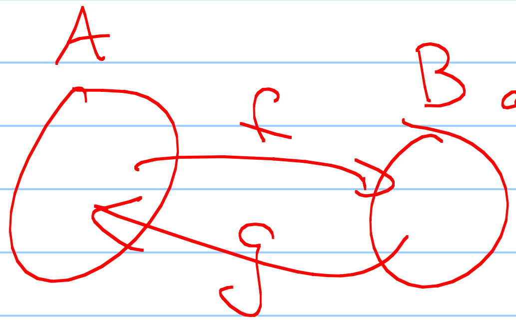
$f \circ g = i_B$ and $g \circ f = i_A$.

pf: Suppose f & g are inverses.

By dfn!!

$$f(a) = b \iff g(b) = a$$

$$f \circ g(b) = f(g(b)) = f(a) = b \Longrightarrow f \circ g(b) = i_B(b)$$



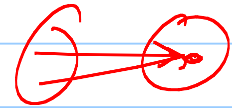
and $g \circ f(a) = g(f(a)) = g(b) = a$

So $g \circ f = i_A$

$\square$

__Thm__: Let A & B be finite sets, with $f: A \to B$.

a) If f is 1-1, then $|A| \le |B|$



b) If f is onto, then $|A| \ge |B|$.

__pf of b__: by contradiction

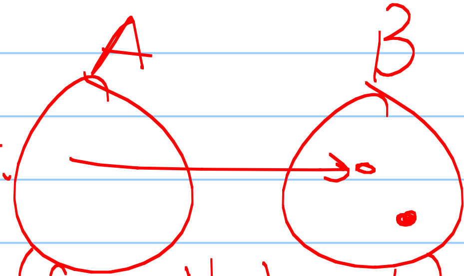
Suppose f is onto.

Assume $|A| < |B|$.

f is onto: $\forall b \in B, \exists a \in A$ s.t. $f(a) = b$.

each a can only map to 1 item in B.

So f can at most "hit" $|A|$ things.



This means something in B didn't get hit, since $|A| < |B|$

↯ (f can't be onto)

Cor: If $f : A \to B$ is a bijection, then $|A| = |B|$.

(These can be a powerful counting tools.)