Math 135 - Sets & Functions

Note Title 9/13/2012

Announcements

- HW 3 is posted — due next Friday

Russell's paradox

Sets are basic mathematical objects, but be careful of contradictions.

Ex: Let A be the set of sets which do not contain themselves:

$$A = \{ S \mid S \notin S \}$$

For example,

$$\emptyset \notin \emptyset \Rightarrow \emptyset \in A$$

$(\emptyset \subseteq \emptyset$, not $\emptyset \in \emptyset)$

$A = \{ S \mid S \notin S \}$

is A in here?

Question: Is $A \in A$?

Every element in A is a set which does not contain itself, so $A \in A$ is impossible.

But then $A \notin A$, so A is a set which doesn't contain itself

$\Rightarrow A \in A$ by definition.

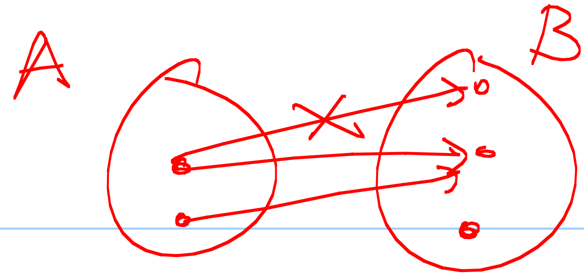
Solution:
To keep mathematics whole, we declare that A is not a set.

Formal set theory starts with the assumption that $\emptyset$ & $\mathbb{N}$ are sets & provides rules to build sets.

Ex: If S is a set, $P(S)$ is a set.

In this class, most of our sets will be "legal", so we won't worry too much. (See Naive Set Theory by Halmos or go take logic if you're interested.)

(2.3)



Functions

Let A and B be sets. A <u>function</u> from A to B, written $f: A \to B$, is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ where $a \in A$, $b \in B$.

A is called the <u>domain</u> of f, and B is the <u>co-domain</u>.

<u>range</u> $\subseteq B$
includes anything "hit" by f

Examples

① $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x + 1$

$f(3) = 4$

② Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Domain: $\{T, F\} \times \{T, F\}$

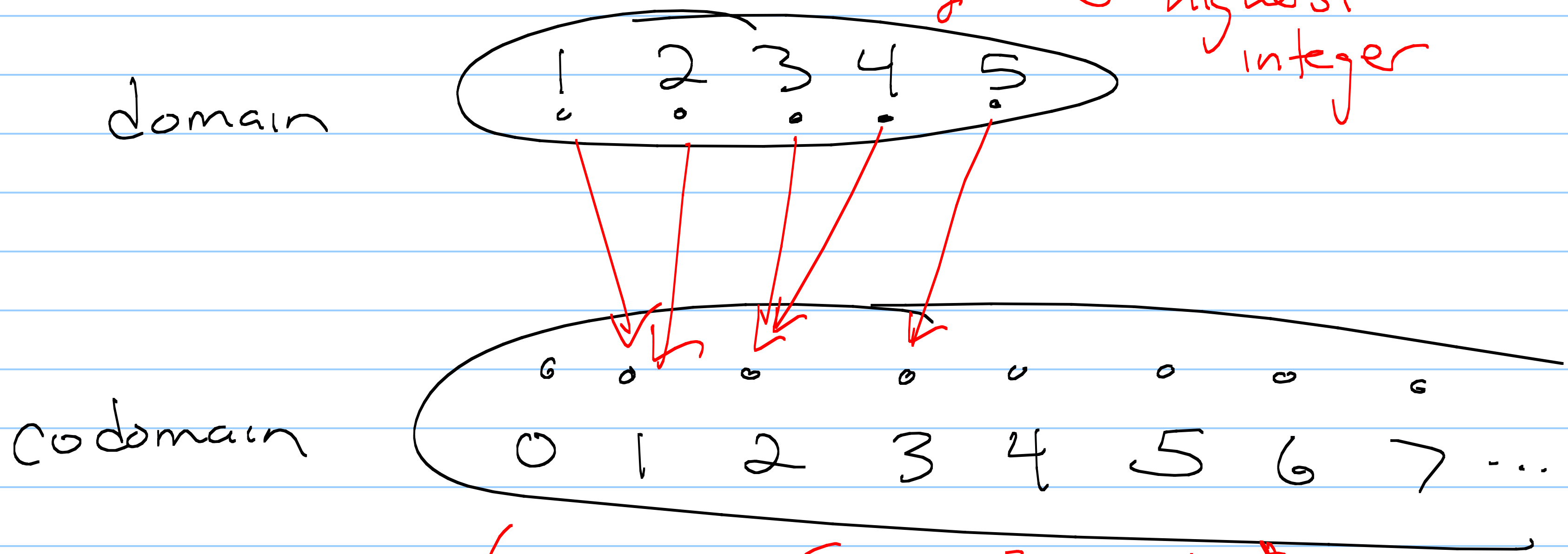
codomain: $\{T, F\}$

p	q	$p \to q$

domain co-domain

(3) $f: \{1, 2, 3, 4, 5\} \longrightarrow \mathbb{N}$

$f(x) = \left\lceil \frac{x}{2} \right\rceil$ ← ceiling: go ~~to~~ next highest integer



(range: $\{1, 2, 3\} \subseteq \mathbb{N}$)

$P(X) = \{\phi, \{a\}$

domain codomain

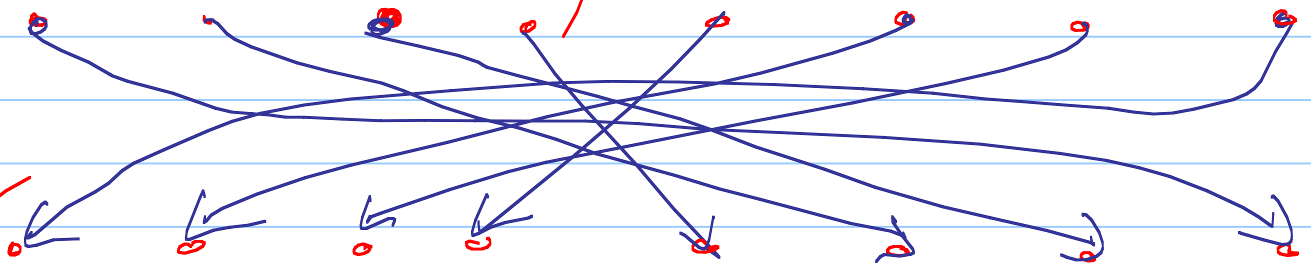
(4) Let $X = \{a, b, c\}$ and $c : \underset{\text{domain}}{P(X)} \to \underset{\text{codomain}}{P(X)}$ be the function

$$c(A) = X - A$$

$$c(\{b\}) = X - \{b\} = \{a, c\}$$

$$c(\phi) = X - \phi$$

domain $P(X)$: ϕ $\{a\}$ $\{b\}$ $\{c\}$ $\{a,b\}$ $\{b,c\}$ $\{a,c\}$ $\{a,b,c\}$



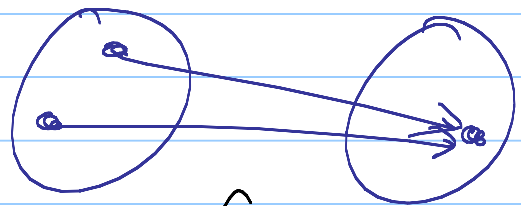
codomain $P(X)$: ϕ $\{a\}$ $\{b\}$ $\{c\}$ $\{a,b\}$ $\{b,c\}$ $\{a,c\}$ $\{a,b,c\}$

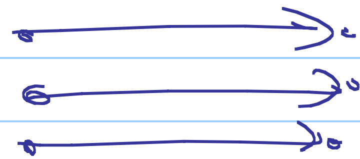
Dfn: A function $f: A \to B$ is one-to-one (1-1) (or injective) if and only if $f(a) = f(b)$ implies $a = b$.

Such a function is an injection.

Logic notation: $\forall a,b \quad f(a)=f(b) \Rightarrow a=b$.

No





So for these functions, no element in B has more than 1 element of A mapping to it.

Ex: $f:\{a,b,c,d\} \longrightarrow \{1,2,3,4,5\}$

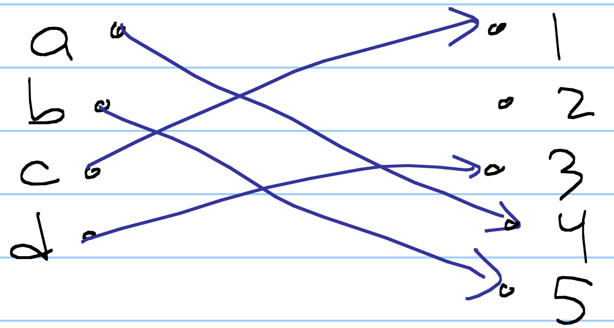
with:

$f(a)=4$

$f(b)=5$

$f(c)=1$

$f(d)=3$



Is this 1-1?

yes

Ex: $f:\mathbb{Z} \to \mathbb{Z}$, with $f(x)=x^2$ Not 1-1:

$f(1)=1$ $f(-1)=(-1)^2=1$

$f(2)=4$ $f(1)=f(-1)$ and $1 \neq -1$

Ex: $f:\mathbb{N} \to \mathbb{N}$, with $f(x)=x^2$

yes?

Prove that $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x+1$ is injective.

1-1

pf: Need to show $\forall x \forall y$, if $f(x) = f(y)$, then $x \overset{?}{=} y$.

Spps $f(a) = f(b)$.

$\Rightarrow a+1 = b+1$

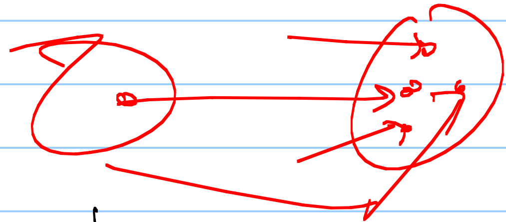
Subtract 1 from both sides:

$\rightarrow a = b$

$\square$

Dfn: A function is called onto (or surjective) if & only if for every $b \in B$, there is an $a \in A$ with $f(a) = b$.

In logic: $\forall b \in B, \exists a$ s.t. $f(a) = b$.

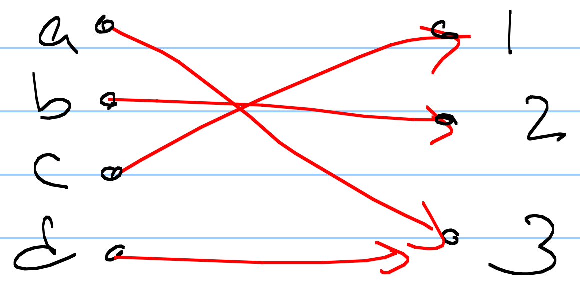


or range = codomain

So for surjections, every element of B must be an "output" of f.

Examples: ① $f: \{a, b, c, d\} \to \{1, 2, 3\}$

$f(a) = 3$
$f(b) = 2$
$f(c) = 1$
$f(d) = 3$



Onto? Yes

1-1? No: $f(a) = f(d)$ but $a \neq d$

② $f: \mathbb{Z} \to \cancel{\mathbb{Z}}\ \mathbb{N}$, $f(x) = x^2$

onto? No: no x exists s.t. $f(x) = -1$

$\forall x$, $f(x) \geq 0$ so nothing has $f(x) = -1$

Not onto — can't hit 2

b/c $\sqrt{2}$ is irrational, so $\notin \mathbb{Z}$

③ $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x+1$

onto? Yes

$y \in \mathbb{Z}$

let $x = y-1$

Dfn: A function is a bijection if it is both 1-1 and onto.

Ex: $f(x) = x+1$

$c(A) = X - A$ (powerset one a few slides ago)

Ex: Is $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x+1$, a bijection? Yes: proof

1-1? Spps $f(a) = f(b)$

Yes: $\Rightarrow 2a+1 = 2b+1$

$\Rightarrow 2a = 2b$

$\Rightarrow a = b$ ✓

onto: Take $y \in \mathbb{R}$

$2x+1 = y$

$2x = y-1$

$x = \frac{y-1}{2} \in \mathbb{R}$

Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2+1$?

1-1? $f(-1) = f(1)$

onto? no