Math 135 — Recursion Trees

Note Title 11/2/2012

Announcements

- HW update: 4c - give general form
  full solution: extra credit
- Test in 1 week
  (come & pick up sample
  review on Wed!)

Divide & Conquer Recurrences (8.3)

$$A(n) = a\,A\left(\frac{n}{b}\right) + g(n)$$

- Non-linear, so characteristic equation method doesn't apply.
- Can unroll, but messy
- Solution: Recursion trees

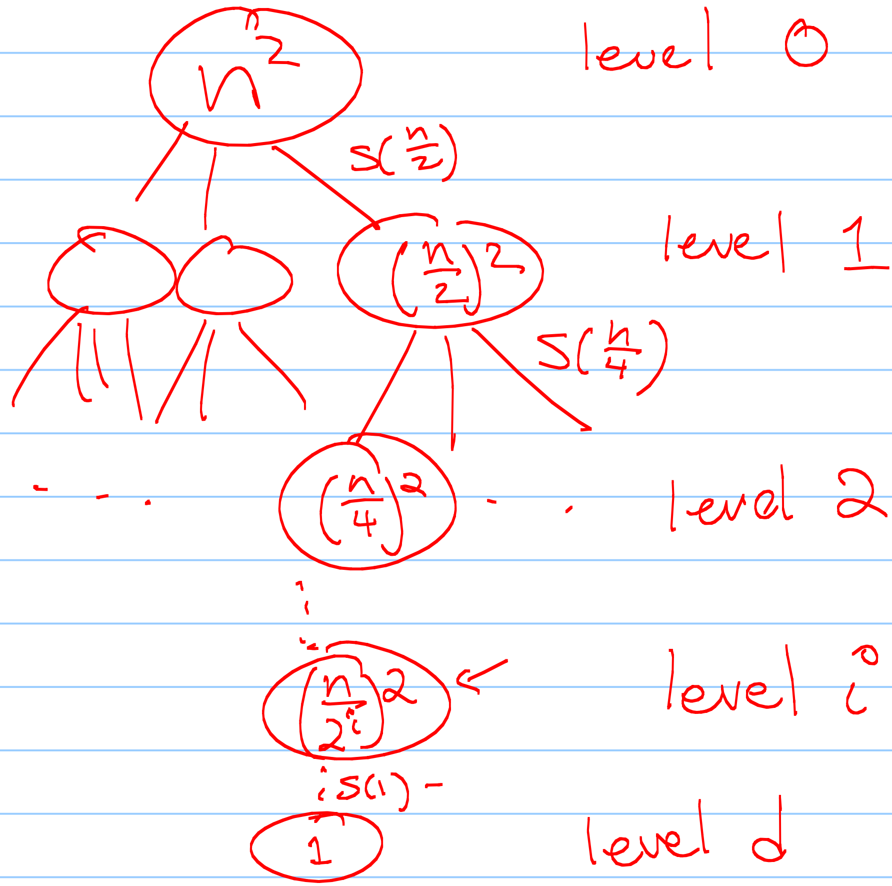
$S(k) = 3S(\frac{k}{2}) + k^2$

Example: $S(n) = \underline{3}S(\frac{n}{2}) + \underline{n^2}$, $S(1) = 1$

$3^1 = 3$ nodes

$3^2 = 9$ nodes

3^i nodes →



$\frac{n}{2^d} = 1$

$\Rightarrow d = \log_2 n$

Θ : both O & Ω

$(2^i)^2 = 2^{2i} = (2^2)^i$

So $S(n) =$ work in tree

$$= \sum_{i=0}^{d} (\text{\# nodes})(\text{work per node})$$

$$= \sum_{i=0}^{\log_2 n} 3^i \left(\frac{n}{2^i}\right)^2 = \sum_{i=0}^{\lg n} 3^i \cdot \frac{n^2}{4^i}$$

$$= n^2 \sum_{i=0}^{\lg n} \left(\frac{3}{4}\right)^i < n^2 \underbrace{\sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i}_{= \frac{1}{1-\frac{3}{4}}} = 4n^2$$

$\Rightarrow S(n) = O(n^2)$

Also show $\Omega(n^2)$:

$$S(n) = n^2 \sum_{i=0}^{\log_2 n} \left(\frac{3}{4}\right)^i > n^2 \cdot \left(\frac{3}{4}\right)^0 = n^2$$

so $S(n) > n^2 \Rightarrow \Omega(n^2)$

and $S(n) < 4n^2 \Rightarrow O(n^2)$

so $S(n) = \Theta(n^2)$

$$V(k) = 2V\left(\frac{k}{4}\right) + k^3$$

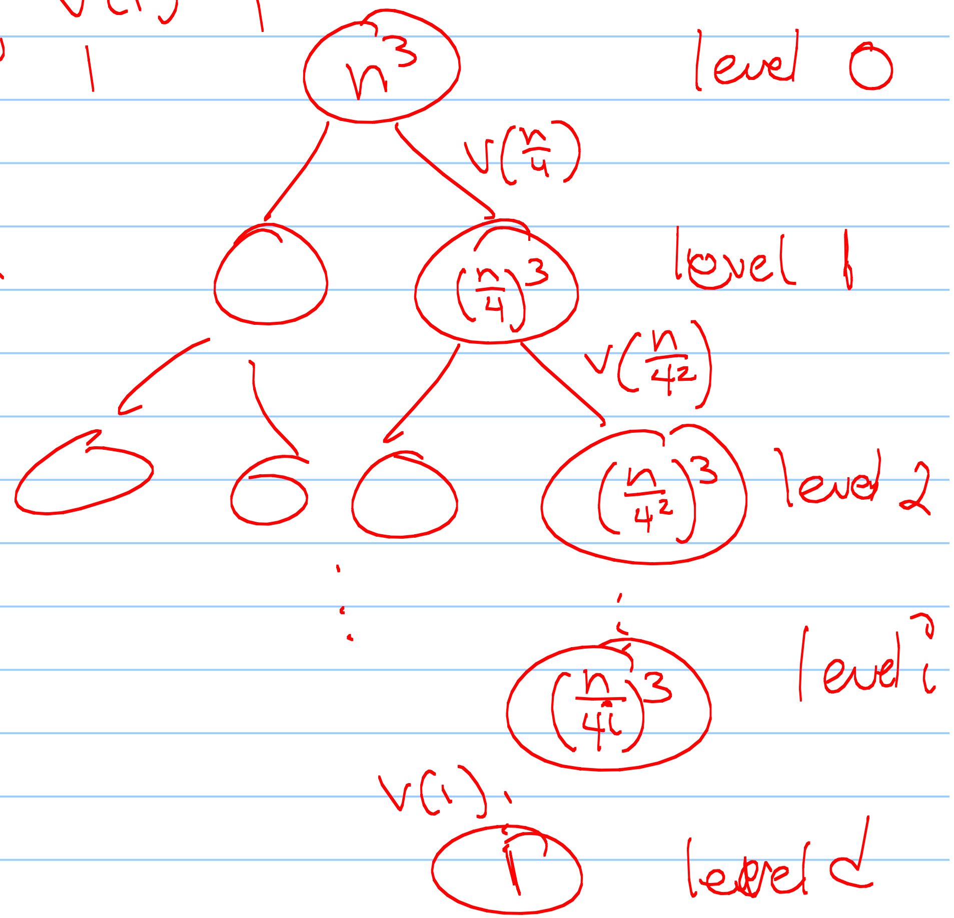
Ex: $V(n) = 2V\left(\frac{n}{4}\right) + n^3$, $V(1) = 1$

2

4

⋮

2^i nodes



$$\frac{n}{4^d} = 1$$

$$n = 4^d$$

$$\log_4 n = d$$

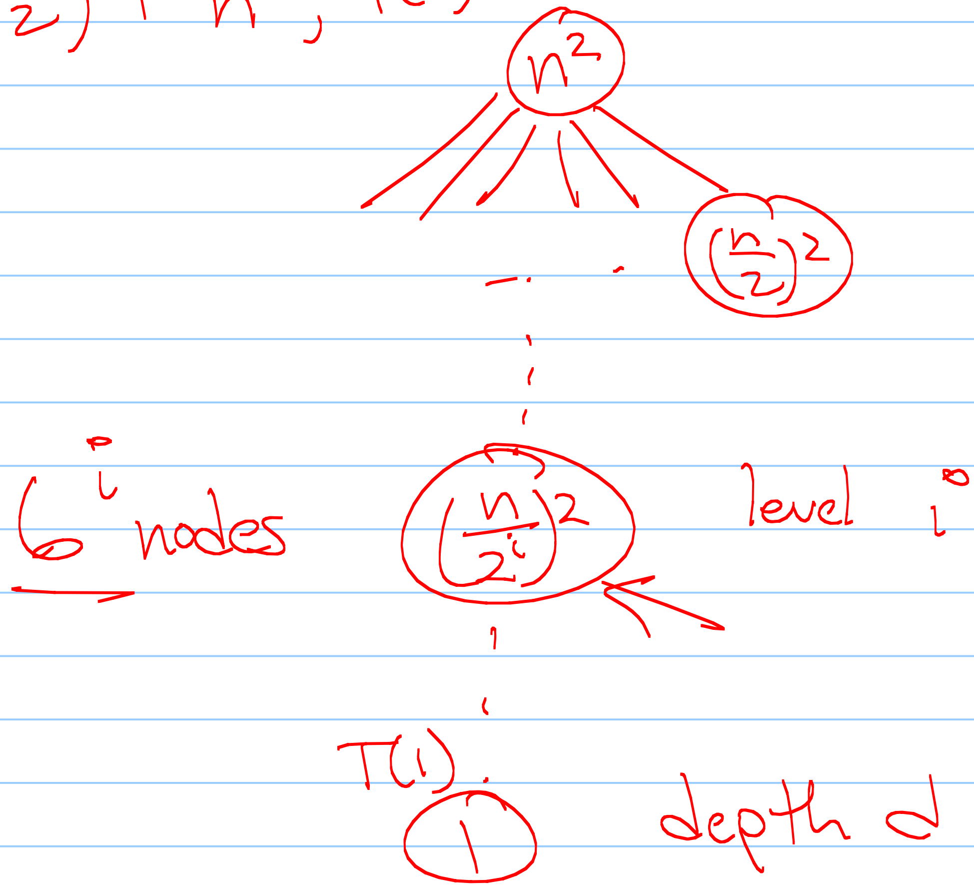
$$V(n) = \sum_{i=0}^{?} (\#\text{nodes})(\text{work per node})$$

$$= \sum_{i=0}^{\log_4 n} (2^i)\left(\frac{n}{4^i}\right)^3$$

$$= n^3 \sum_{i=0}^{\log_4 n} \frac{2^i}{64^i}$$

Same sums $\Rightarrow V(n) = \Theta(n^3)$

Ex: $T(n) = 6T\left(\frac{n}{2}\right) + n^2, \; T(1) = 1$



$\frac{n}{2^d} = 1$

$\Rightarrow d = \log_2 n$

$$T(n) = \sum_{\text{levels}} (\#\text{nodes})(\text{work per node})$$

$$= \sum_{i=0}^{\log_2 n} (6^i)\left(\frac{n}{2^i}\right)^2$$

$$= n^2 \sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i = n^2 \left(\frac{\left(\frac{3}{2}\right)^{\log_2 n+1} - 1}{1/2}\right)$$

$$\left(\frac{3}{2}\right)^{\log_2 n} = n^{\log_2 (6)}$$

Ick. But – there is a pattern here!

$$f(n) = a f\left(\frac{n}{b}\right) + g(n)$$

$$\Rightarrow f(n) = \sum_{i=0}^{\log_b n} a^i g\left(\frac{n}{b^i}\right)$$

These may look like our series/summation formulas!

Recap:

$$\sum_{k=0}^{m} a \cdot r^k = \frac{a \cdot r^{m+1} - a}{r-1} \quad (\text{if } r \neq 0)$$

and

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{if } |x| < 1$$

Master Thm:

Let f satisfy $f(n) = af\left(\frac{n}{b}\right) + \Theta(n^k)$ where $a \geq 1$, b is an integer > 1, and k is a real number ≥ 0.

Then:

$$f(n) = \begin{cases} O(n^k) & \text{if } a < b^k \\ O(n^k \log_b n) & \text{if } a = b^k \\ \underbrace{O(n^{\log_b a})} & \text{if } \underbrace{a > b^k} \end{cases}$$

$\Theta(n^k)$

How to use:

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Here, $a = 2$

$b = 2$

$k = 1$

$\Big\}\ b^k = 2^1 = 2$

So: $a = b^k$

$$\Rightarrow T(n) = \Theta(n^1 \log_2 n) = \Theta(n \lg n)$$

Ex:

$$T(n) = 3T\left(\frac{n}{2}\right) + n$$

Here, $a = 3$

$b = 2$

$k = 1$

$\left\{ 2^1 = b^k \right.$

So: $3 > 2^1$

So $a > b^k$

So $T(n) = O(n^{\log_2 3})$

$$aT\left(\frac{n}{b}\right)$$

Ex: $T(n) = T\left(\frac{3n}{4}\right) + n^2$

$$\left.\begin{aligned} a &= 1 \\ b &= 4/3 \\ k &= 2 \end{aligned}\right\} \quad b^k = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$$

So: $1 = a < b^k = \frac{16}{9}$

So $T(n) = O(n^2)$

$$T(k) = \sqrt{k}\, T(\sqrt{k}) + k$$

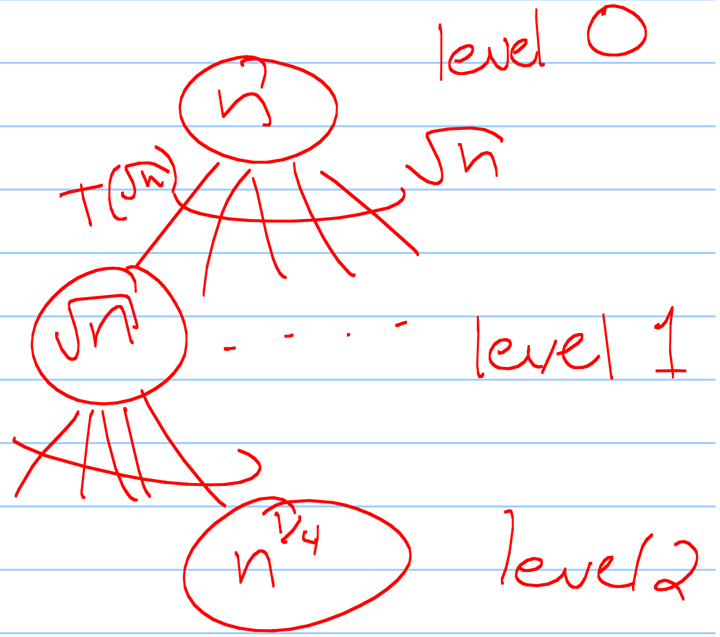
$\sqrt{n} = n^{1/2}$

When Master Thm <u>fails</u>:

$$T(n) = \sqrt{n}\, T(\sqrt{n}) + n$$

$\underline{n^{1/2}} \rightarrow$

$n^{\frac{1}{2}} \cdot n^{\frac{1}{4}}$ nodes $\rightarrow$

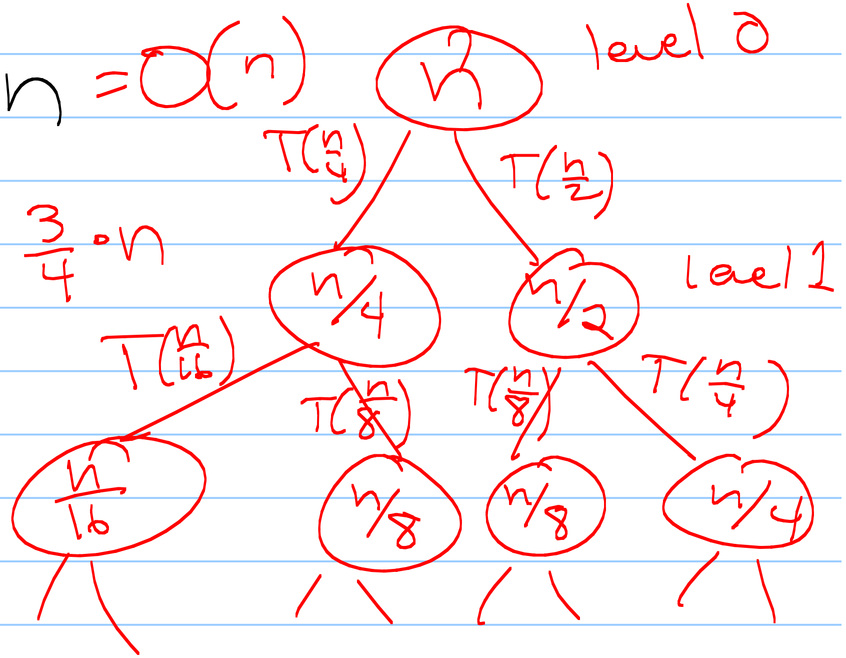


next time — continued

$T(k) = T(\frac{k}{2}) + T(\frac{k}{4}) + k$

rec tree:

$T(n) = T(\frac{n}{4}) + T(\frac{n}{2}) + n = O(n)$



total work: $\frac{3}{4} \cdot n$

total work: $\frac{9}{16} \cdot n$

$$\sum_{\text{all levels } i} \left(\frac{3}{4}\right)^i \cdot n = n \sum_{\text{levels } i} \left(\frac{3}{4}\right)^i < 4n$$