Math 135 – Counting (pt. 3)

Note Title 11/15/2012

Announcements

- Turn in HW
- Next HW – up this weekend
  due Wed. after break

Permutations 6.3 (?)

How many ways are there to list r distinct elements from a set of size n?

Dfn: This is $P(n,r)$. P^n_r

Rule of product:

$$\underline{n} \cdot \underline{(n-1)} \cdot \underline{(n-2)} \; \underline{\quad} \cdot \underline{\quad} \cdot \underline{(n-r+1)}$$

r distinct elements

Formula: $P(n,r) = \dfrac{n!}{(n-r)!}$

Ex: Suppose we have 8 runners & will award 3 medals (gold, silver & bronze).

Assuming no ties, how many different possible ways to award?

rule of product: $\underline{8} \cdot \underline{7} \cdot 6$

$$P(8,3) = \frac{8!}{5!}$$

Ex: How many different permutations of the alphabet contain the string "ABC"?

One letter

23 other letters

$\Rightarrow n = 24$

$r = 24$

$$P(24, 24) = 24! = \frac{n!}{(n-r)!} = \frac{24!}{0!}$$

Combinations

How many different ways are there to choose r elements out of n?

order does not matter

Notation: $C(n,r) = \binom{n}{r} =$ "n choose r"

book

everyone else

Ex: How many different ways to choose 2 elements from $\{1, 2, 3, 4, 5\}$? $n = 5$, $r = 2$

Ans: $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$

$\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$

$\{3, 4\}$, $\{3, 5\}$

$\{4, 5\}$

$$\binom{n}{r} = \binom{5}{2} = 10$$

Thm: $P(n,r) = \binom{n}{r} \cdot P(r,r)$

pf: Via "combinatorial arguement".

LHS: $P(n,r)$: listing r things from a set of size n (by dfn)

RHS: Rule of product

First, choose the r elements (by dfn) $= \binom{n}{r}$

Second, order the r elements

This gives a nice formula!

$$P(n,r) = \binom{n}{r} P(r,r)$$

$$\Rightarrow \binom{n}{r} = \frac{P(n,r)}{P(r,r)}$$

$$= \frac{\frac{n!}{(n-r)!}}{\frac{r!}{0!}} = \frac{n!}{(n-r)!\,r!}$$

Ex: How many different poker hands are there?

(52 different cards, 5 cards in a hand)

$$52 = n \qquad 5 = r$$

$$\binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!}$$

Ex: How many bit strings of length 5 have exactly 3 ones?

$$\binom{5}{3} = \frac{5!}{3!\,2!}$$

choose 3 spots for a 1

Follow-up: How many bitstrings of length n have exactly r ones?

$$\binom{n}{r}$$

Combinatorial proof:

A proof that uses a counting arguement to prove that two formulas count the same thing (& so must be equal)

Ex: $\binom{n}{r} = \binom{n}{n-r}$

counts # of bitstrings of length n with r 1's.

counts # of bit strings of length n with n-r 0's

These count the same strings 2 different ways.

Thm: $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$

pf: LHS: Number of ways to choose a committee of size $r+1$ from a group of $n+1$ people

RHS: Use rule of sum: Pick someone from group of $n+1$. Call him Bob.

Two disjoint possibilities: Bob is on committee or not.

If on, choose $\binom{n}{r}$ others.

If not, choose $\binom{n}{r+1}$ others.

Permutations with repetition

How many strings of length r can be formed from the English alphabet?

Rule of product: $\underline{26} \cdot \underline{26}\ \underline{\ -\ }\ \underline{\ -\ }\ \underline{\ -\ }\ \underline{\ -\ } \dots \underline{\ -\ }\ \underline{26} = 26^r$

r spots

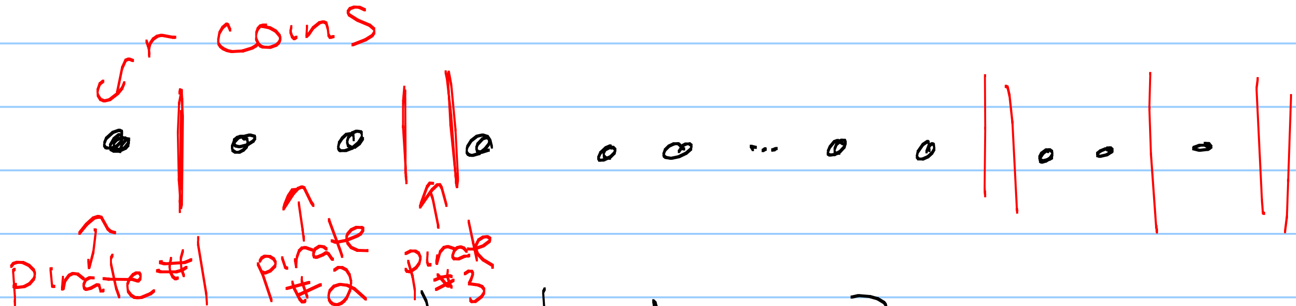
n possibilities: n^r

[Note: <u>Not</u> $P(26, r) = 26 \cdot 25 \cdot 24 \cdots (26 - r + 1)$]

Combinations revisited

How many ways are there to distribute r identical gold coins among n pirates?

Trick: Place coins in a row:



How can we divide them?

put $n-1$ dividers in

In total, have $\underbrace{r}_{\text{coins}} + \underbrace{(n-1)}_{\text{bars}}$

(so n piles)

Need to choose r spaces for the coins – rest will be bars

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r coins
$n-1$ dividers

$$\binom{r+n-1}{r}$$