Note Title 11/14/2012

Math 135 – Counting (pt. 2)

Last Time : Ch 6?

1. Rule of sum

   If B & C are disjoint sets and $A = B \cup C$, then

   $$|A| = |B| + |C|$$

2. Rule of product

   Suppose a set can be formulated as a sequence of k choices.

   Then if there are n_1 ways to make first choice, n_2 to make second, etc.

   $$|A| = n_1 \cdot n_2 \cdots n_k$$

__Example__ (from last time)

In one version of the programming language BASIC, variables could be 1 or 2 alphanumeric characters.

- Had to begin with letter
- 5 reserved forbidden keywords
- No distinguishing upper/lower case

How many variables?

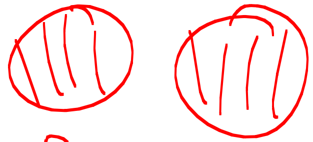
Rule of sum: #1 char + #2 char − 5

#1 char : 26

#2 char : $\underline{26 \cdot 36}$

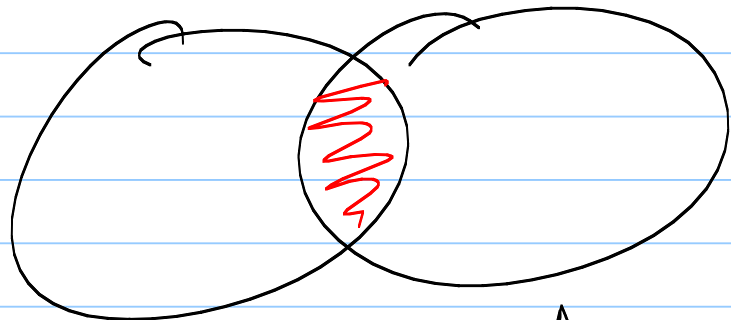
__Ans__ : $26 + 26 \cdot 36 - 5$

<u>Principle of Inclusion / Exclusion</u>

$A =$  $B \cup C$

- generalizes the rule of sum

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$



A_1 A_2

Ex: How many bitstrings of length n either start with a 1 or end with 00?

$$\underbrace{\underline{2}\cdot\underline{2}\cdot\underline{2}\ \_\ \_\ \cdots\ \_\ \underline{2}}_{n \text{ spots}} = 2^n$$

Start w/ a 1: 2^{n-1}

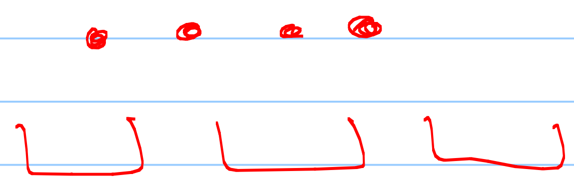
End w/ a 00: 2^{n-2}

Start w/ a 1 and end w/ 00: 2^{n-3}

Ans: $2^{n-1} + 2^{n-2} - 2^{n-3}$

Next Section: The Pigeonhole Principle 6.2

Thm: If k is a positive integer and $k+1$ balls are placed into k boxes, then some box has 2 or more balls.



pf: contrapositive:

Say each box has at most 1 ball,

Then $\leq k$ balls total.

Examples

- A function from a set with $k+1$ elements to a set with k elements is not 1-1.

  boxes: elements in codomain
  balls: elements in domain
  function puts balls in boxes

- In any group of 367 people, 2 have the same birthday

  boxes: days of year
  balls: people
  Birthday is box each person is put in.

S : 10

More Complex

Show that for every integer n, there is a multiple of n written with only 0's and 1's (in decimal).

boxes: remainder when divided by n

n boxes: $0, 1, \ldots, n-1$

balls: integers with only 1's:

1

11

111

⋮

$11\cdots1$ ← $n+1$ 1's

So 2 of the 1's strings have same remainder

$8 \bmod 5$
$13 \bmod 5$

$13 - 8 = 5$

same remainder mod n

$\Rightarrow x - y$ is divisible by n

$1111\ldots1 - 11\ldots1 =$ string of 1's + 0's

Ex: $n = 8$

1
11
111
1111
11111
111

rem: 1
rem: 3
rem: 7
rem: 7

$$\begin{array}{r} 13 \\ 8\,\overline{)\,111} \\ \underline{8} \\ 31 \\ \underline{24} \\ 7 \end{array} \qquad \begin{array}{r} 1 \\ 8\,\overline{)\,11111} \\ \underline{8} \end{array}$$

$1111 - 111 = 1000$ is div by 8

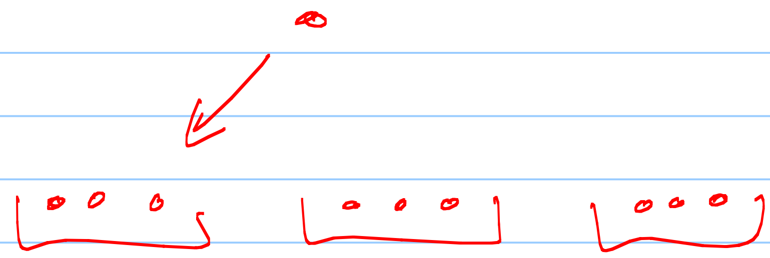
(For any 2 numbers that have same remainder, $x - y$ is divisible by n.)

Generalized Pigeonhole Principle

If N balls are placed into k boxes, then there is a box containing at least $\lceil N/k \rceil$ balls.

Ex: 10 balls
3 boxes

$$\left\lceil \frac{10}{3} \right\rceil = 4$$



Ex : Among 100 people, how many must have been born in the same month?

boxes = months
balls = people

$$\left\lceil \frac{100}{12} \right\rceil = 9$$

Ex: How many cards must we select from a standard 52 card deck to be sure 3 of them are the same suite?

boxes = suites (4 of them)
balls = N cards

$$\left\lceil \frac{N}{4} \right\rceil = 3$$

Need 9 cards

Ex: During a month with 30 days, a baseball team plays at least 1 game a day, but no more than 45 total.

Show there is a period of consecutive days where the team plays **exactly** 14 games.

Let a_j = number of games played on or before day j in the month.

$1 \le a_1 \le a_2 \le a_3 \le \dots \le a_{30} \le 45$

Make another list: $15 \le a_1+14, a_2+14, \dots, a_{30}+14 \le 59$

Let a_j = number of games played on or before day j in the month.

$1 \leq a_1 \leq a_2 \leq a_3 \leq \dots \leq a_{30} \leq 45$

Make another list: $15 \leq a_1+14, a_2+14, \dots, a_{30}+14 \leq 59$

Consider these 60 numbers. ← balls

They range from 1 to 59. ← boxes

pigeonhole ⟹ 2 are equal.

$a_i = a_k + 14$

Between day k and day i, play exactly 14 games.