Math 135 - End of Recurrences
Intro to Counting

Note Title 11/5/2012

Announcements

- Turn in HW 7
- HW6 - graded
- Test Friday, review in class Wed.
- Office Hours: Wed. 1-2 (or 9-10)
  Thurs. 1-2
  (Canceled on Friday)

Master Thm:

Let f satisfy $f(n) = af\left(\frac{n}{b}\right) + \Theta(n^k)$
where $a \geq 1$, b is an integer > 1,
and k is a real number ≥ 0.

Then:

$$f(n) = \begin{cases} O(n^k) & \text{if } a < b^k \\ O(n^k \log_b n) & \text{if } a = b^k \\ \underbrace{O(n^{\log_b a})} & \text{if } \underbrace{a > b^k} \end{cases}$$

Final word on Master Thm

Recursion tree:

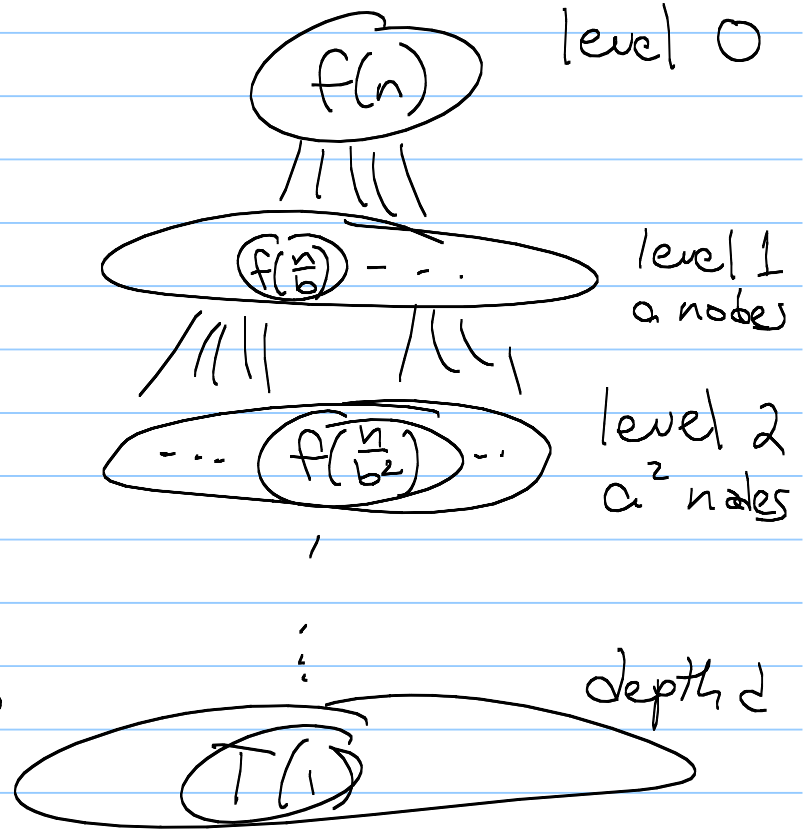
$$T(n) = a\,T\!\left(\frac{n}{b}\right) + f(n)$$

$$T\!\left(\frac{n}{b}\right) = a\,T\!\left(\frac{n}{b^2}\right) + f\!\left(\frac{n}{b}\right)$$

$$T\!\left(\frac{n}{b^2}\right) = a\,T\!\left(\frac{n}{b^3}\right) + f\!\left(\frac{n}{b^2}\right)$$

$$\vdots$$

$$\frac{n}{b^d} = 1 \Rightarrow d = \log_b n$$



$$T(n) = \sum_{i=0}^{\log_b n} a^i f\left(\frac{n}{b^i}\right) = f(n) + af\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + \cdots + a^{\log_b n} f(1)$$

Master thm just says this is increasing or decreasing geom. series.

<u>Case 1</u>: increasing → fraction is > 1

<u>Case 2</u>: amount on each "level" is same

<u>Case 3</u>: decreasing → fraction is < 1

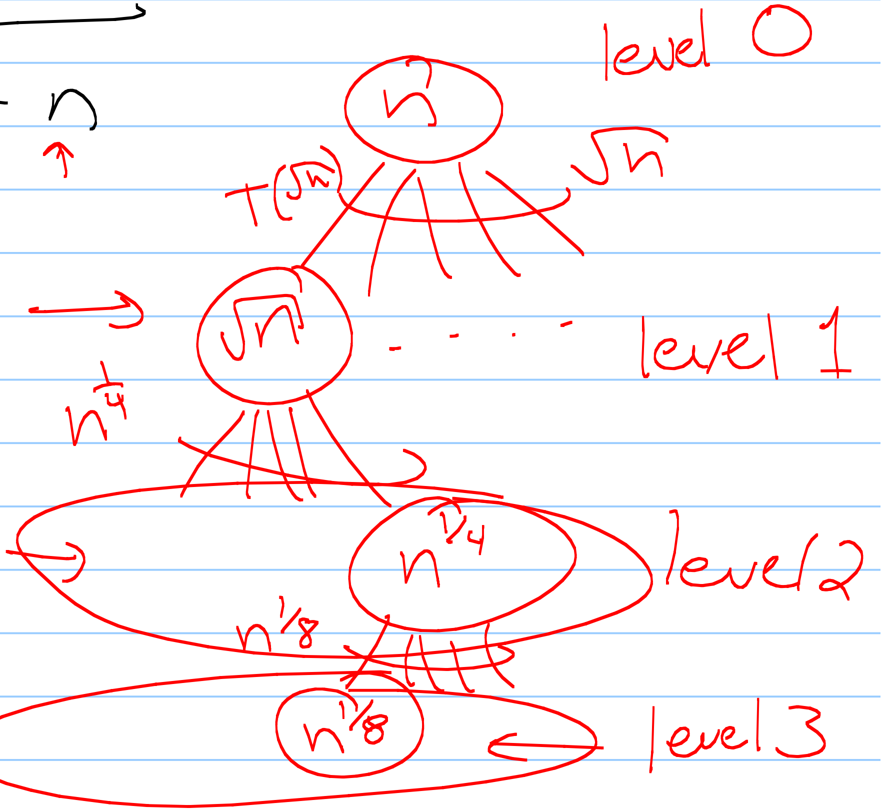
$$T(k) = \sqrt{k}\,T(\sqrt{k}) + k$$

$\sqrt{n} = n^{1/2}$

When Master Thm <u>fails</u>:

$$T(n) = \sqrt{n}\,T(\sqrt{n}) + n$$

$\underline{n^{1/2}}$



$n^{3/4} = n^{\frac{1}{2}} \cdot n^{\frac{1}{4}}$ nodes

$n^{3/4} \cdot n^{1/8} = n^{7/8}$ nodes on level

Cont:

$n^{\frac{1}{2^d}} = 2$

$\log_2\left(n^{\frac{1}{2^d}}\right) = \log_2 2$

$\frac{1}{2^d} \log_2 n = 1$

$\log_2 n = 2^d$

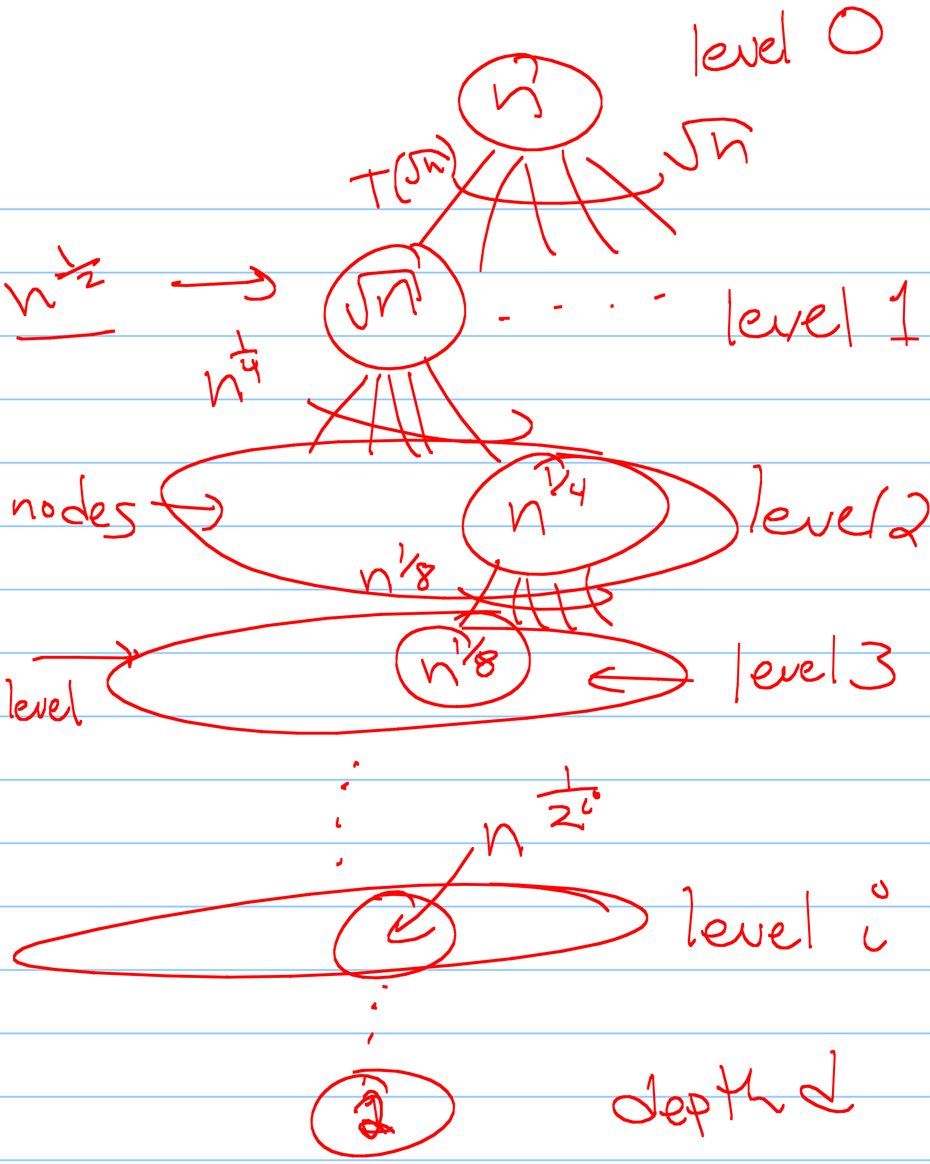
$\log\log n = d$

$n^{\frac{1}{2}}$

$n^{3/4} = n^{\frac{1}{2}} \cdot n^{\frac{1}{4}}$ nodes

$n^{3/4} \cdot n^{1/8} = n^{7/8}$ nodes on level

$n^{1-\frac{1}{2^i}}$ nodes



$$\sum_{i=0}^{d} (\#\text{nodes})(\text{work per node})$$

$$= \sum_{i=0}^{\lg\lg n} \left(n^{1-\frac{1}{2^i}}\right)\left(n^{\frac{1}{2^i}}\right)$$

$$= \sum_{i=0}^{\lg\lg n} n = \Theta(n \lg\lg n)$$

Counting: Ch 6

(Ch. 6.1)

2 basic principles

① Rule of Sum

② Rule of Product

Rule of Sum

If B & C are disjoint sets and $A = B \cup C$, then

$$|A| = |B| + |C|$$

We split A into non-overlapping subsets, so can just sum sizes of B & C.

Ex: Need a math representative for a committee. There are 37 students & 12 faculty available.

Total possible choices: $37 + 12 = 49$

$(1,2)$ $(n-1,n)$

$(1,1)$

<u>Ex</u>: $A = \{(x,y) \in \{1,2,\dots,n\}^2 : x=4 \text{ or } x=5\}$

Recall: $\{1,2,\dots,n\}^2$ is set of ordered pairs (x,y) with $1 \le x \le n$ & $1 \le y \le n$.

<u>Here</u>: $|A| = |\{(4,y) \mid 1 \le y \le n\}|$
$+ |\{(5,y) \mid 1 \le y \le n\}|$

$= n + n = 2n$

Rule of Product

Suppose a set can be formulated as a sequence of k choices.

Then if there are n_1 ways to make first choice, n_2 to make second, etc.

$$|A| = n_1 \cdot n_2 \cdots n_k$$

Ex: How many binary strings of length n?

00
01
10
11

$= 4$

000
001
010
011
100
101
110
111

8

$$2^n = \underline{2} \cdot \underline{2} \cdot \underline{2} \, \underline{\ \ } \cdot \underline{\ \ } \cdot \underline{\ \ } \cdot \underline{2}$$

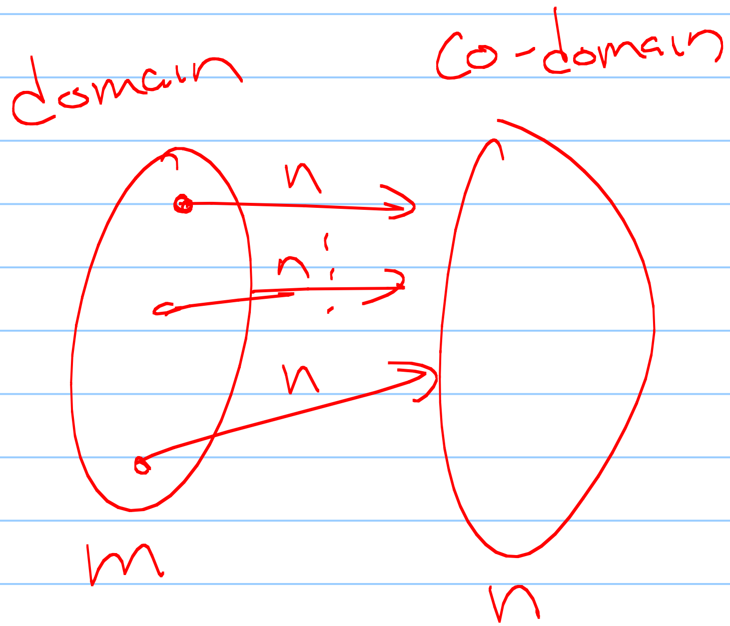
n spots

Ex: Chairs in an auditorium will be labeled with a letter & a positive integer ≤ 100.

How many chairs are possible?

$$\underset{\text{letter}}{\underline{26}} \quad \underset{\text{number}}{\underline{100}} = \boxed{2600}$$

Ex: How many different functions from a set with m elements to a set with n elements?



For each thing in domain, choose "target" in codomain.

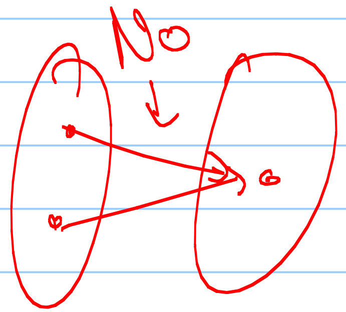
→ Each item has n choices.

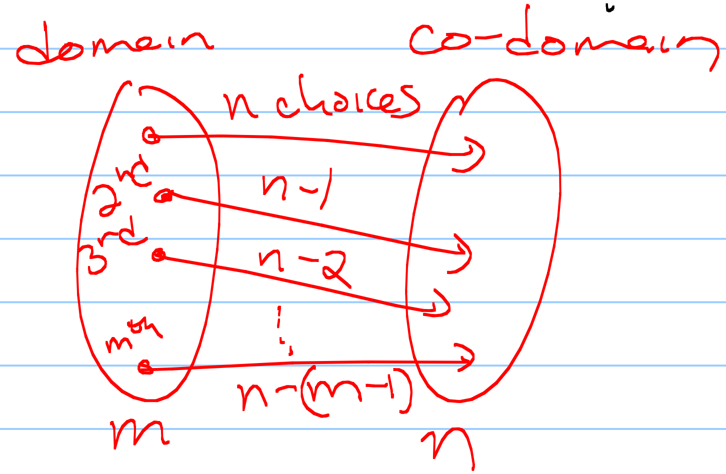
m arrows, n choices each

$\Rightarrow \underbrace{n \cdot n \cdot n \cdot \ldots}_{m}$

$\Rightarrow n^m$

Ex: How many functions are there that are one-to-one?





$$\text{total: } n(n-1)(n-2)\cdots(n-m+1)$$

$$= \frac{n!}{(n-m)!}$$

More Complex

In one version of the programming language BASIC, variables could be 1 or 2 alphanumeric characters.

- Had to begin with letter
- 5 reserved forbidden keywords
- No distinguishing upper/lower case

How many variables?

Rule of sum: #1 char + #2 char − 5

#1 char : 26

#2 char : $26 \cdot 36$

Ans: $26 + 26 \cdot 36 - 5$

Ex: Suppose you need a password.
- 6 to 8 characters long
- Uppercase letters or Numbers
- At least one number.

How many are possible?

$P_6 + P_7 + P_8$

P_6: Count all possible ones $\underline{36}\cdot\underline{36}\cdot\underline{36}\cdot\underline{36}\cdot\underline{36}\cdot\underline{36}$

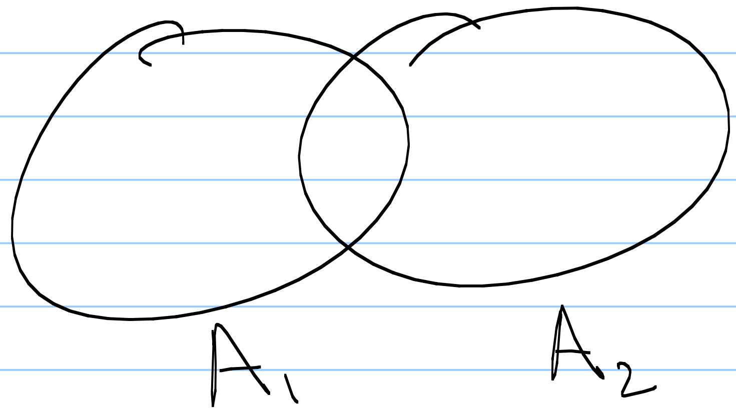
invalid ones: all letters $\underline{26}\cdot\underline{26}\cdot\underline{\ \ }\cdot\underline{\ \ }\cdot\underline{\ \ }\cdot\underline{26}$

$P_6 = 36^6 - 26^6$

Principle of Inclusion/Exclusion

- generalizes the rule of sum

$|A_1 \cup A_2| =$



<u>Ex</u>: How many bitstrings of length n either start with a 1 or end with 00?