Math 135 - Big-O (& Infinite sets)

Note Title 9/20/2012

Announcements

- Please get Worksheets 3 & 4
- Turn in HW3
- Graded HW2 handed back
- HW4 - up later today

Infinite Sets : (ch 2.5)

Dfn: Two sets have the same cardinality $\iff$ there is a bijection from A to B.

Thm: $\mathbb{N}$ & $\mathbb{Z}$ have same cardinality.

~~$\mathbb{N}$ $\mathbb{Z}$~~

~~$0 \mapsto 0$~~

~~$1 \mapsto 1$~~

~~$2 \mapsto 2$~~

~~$\vdots$~~

$1 \to -1$
$3 \to -2$
$5 \to -3$
$7 \to -4$
$9 \to -5$
$\vdots \quad \vdots$

$$n \mapsto \frac{(-1)(n+1)}{2}$$

split even/odd case:

$\mathbb{N} \to$
$0 \to 0$
$2 \to 1$
$4 \to 2$
$6 \to 3$
$8 \to 4$

even $n \mapsto n/2$

Thm: $\mathbb{N}$ & $\mathbb{Q}$ have same cardinality.

(diagonalization)

p/q	0	1	2	3	4	5	... ← p's
1	$0/1$	$1/1$	$2/1$	$3/1$	$4/1$	$5/1$	
2	$0/2$	$1/2$	$2/2$	$3/2$			
3	$0/3$	$1/3$	$2/3$	$3/3$			
4	$0/4$	$1/4$	$2/4$	$3/4$			
5							
⋮							

↑ q's

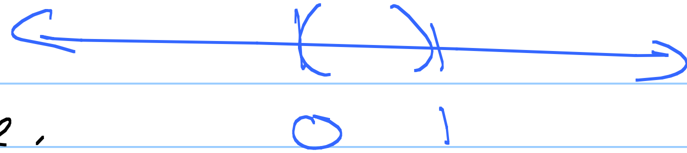
p/q

Q: Are there sets "bigger" than $\mathbb{N}$?

Dfn: A set is countable if there is a bijection $f: \mathbb{N} \to A$ (or if A is finite).

Prev pages show $\mathbb{Z}$ & $\mathbb{Q}$ are countable.

Can show, for example, that $\mathcal{P}(\mathbb{N})$ is not countable.



Thm: $\mathbb{R}$ is not countable.

Actually, we'll show $(0,1) \subseteq \mathbb{R}$ is not countable.

Sketch: Spps we have a bijection $f: \mathbb{N} \to (0,1)$ (for contradiction)

$$
\begin{array}{c|ccccc}
 & d_1 & d_2 & d_3 & d_4 & \cdots \\
\hline
0 & .9 & 6 & 2 & 3 & \cdots \\
1 & a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\
2 & a_{31} & a_{32} & a_{33} & & \\
3 & & & & & \\
\vdots & & & & &
\end{array}
$$

not onto:

Create a real #: $.d_1 d_2 d_3 d_4 \cdots$

$= .8 \cdots (a_{i,i} - 1)$

(can't be hit by f)

Why do we care?

We care about computable things.

What is a computer program?

↳ typing characters

↳ ASCII

↳ 1's & 0's

How many of them are there?

So we can think of a program as "just" a number.

How many functions from $\mathbb{N} \to \{0,1\}$ are there?

x	0	1	2	3	4	5
$f(x)$	0	1	0	0	0	1

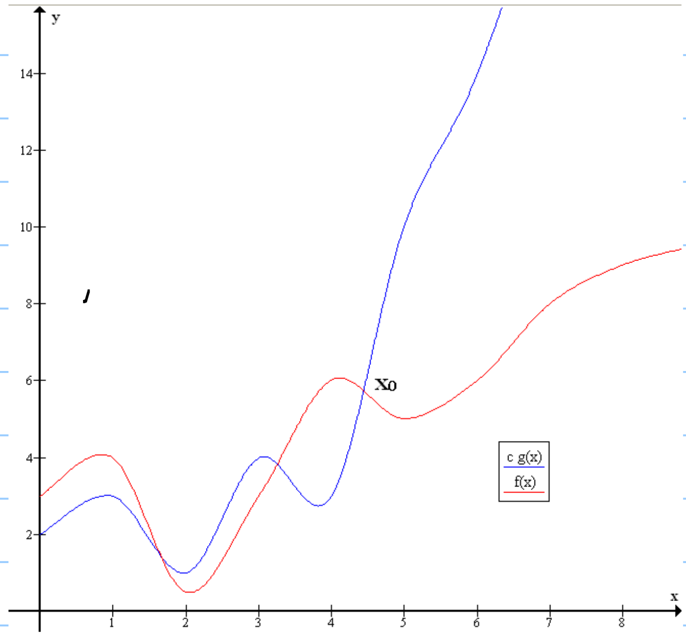
↳ these look like fractions between 0 + 1.

$\Rightarrow$ there are a lot of uncomputable functions!

Big-O Notation – Ch 3.2

How to compare 2 functions?

Which is bigger?



Problem:

It depends on n.

Big-O:

Dfn: Let f & g be functions from $\mathbb{R} \to \mathbb{R}$ (or $\mathbb{Z} \to \mathbb{R}$). We say that:

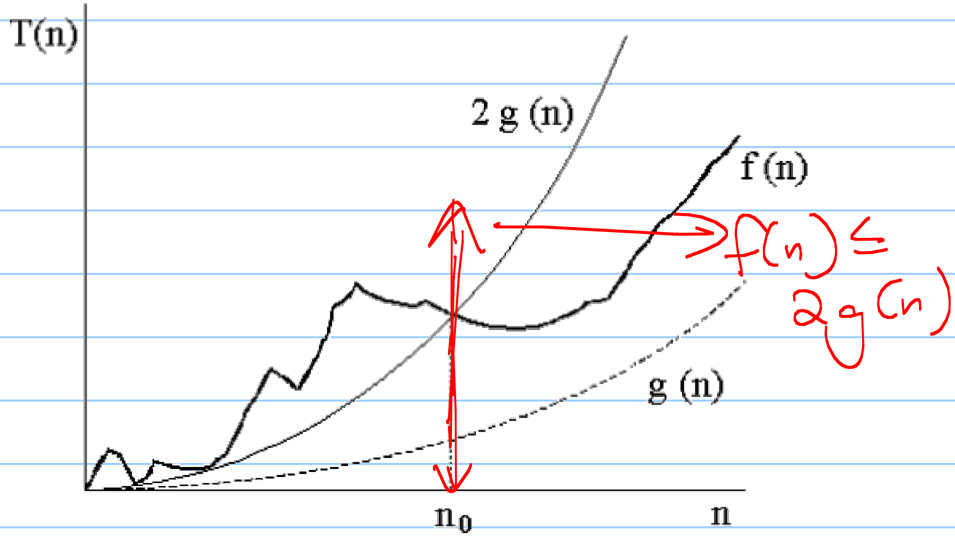
$$f(n) = O(g(n))$$

if there exist constants C & n_0 such that

proved

$$|f(n)| \leq C|g(n)|$$

for all $n > n_0$.



Ex: $f(x) = x^2 + 2x + 1$ is $O(x^2)$

proof: Need to find C and n_0

$$f(x) = x^2 + 2x + 1$$

if $x \geq 1$, then

$$x^2 + 2x + 1 \leq x^2 + (2x)(x) + 1(x^2)$$

$$= x^2 + 2x^2 + x^2$$

$$= 4x^2$$

Let $C = 4$ and $n_0 = 1$

$\blacksquare$

Idea: First select an n_0 that lets you estimate size of $f(n)$ for $n > n_0$.

Then look for a C that makes the inequality work.

So also get:

$f(x) = x^2 + 2x + 1$ is $O(x^3)$

if $x \geq 1$

$$x^2 + 2x + 1 \leq x^2 \cdot x + 2x \cdot x^2 + 1 \cdot x^3$$
$$= 4x^3$$

Let $C = 1$ + $n_0 > 1$

Sometime write $f(x) = O(g(x))$

<u>Not an equality!</u>

- $x^2 + 2x + 1$ is $O(x^2)$
- $x^2 + 2x + 1$ is $O(x^3)$

Really means $f(x) \in \{\text{functions that are } O(g(x))\}$

Ex: Show that $7x^2 = O(x^3)$

one way: if $x > 7$,

then $7x^2 \le x \cdot x^2 = x^3$

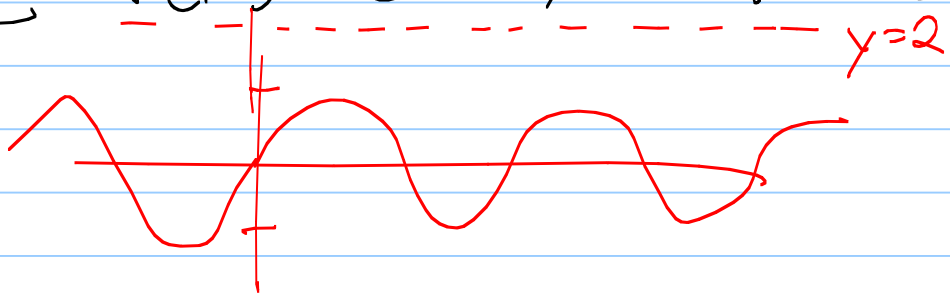
So let $n_0 = 7$ and $C = 1$

another: if $x \ge 1$, then

$7x^2 \le 7x^3$

So let $C = 7$ and $n_0 = 1$

Ex: $f(x) = \sin x$ is $O(1)$.

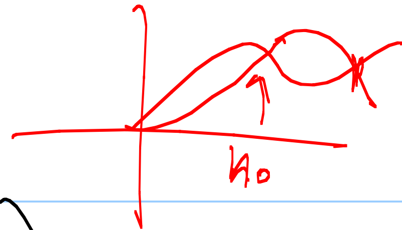


pf: Note, $\sin x \leq 1$

so let $n_0 = 0$ (anything!)

and $C = 2$

∎



Ex: Show that n^2 is not $O(n)$.

pf: Harder: need to show that no constants c & n_0 can exist with $n^2 \le c \cdot n$ for some $n > n_0$.

$$\neg\left(\exists c\, \exists n_0 \ \& \ \forall n > n_0,\ n^2 \le c \cdot n\right)$$

$$= \forall c\, \forall n_0 \ \& \ \exists n > n_0,\quad n^2 > c \cdot n$$

Consider any c & n_0.

pick $n = \max\{c+1, n_0+1\}$

know $n > c$

$$\Longrightarrow \quad n \cdot n > c \cdot n$$

$\square$

Ex: Consider $\sum_{i=1}^{n} i$.

What is a big-O bound? $O(n^2)$

(Two ways to do this.)

1. Use known formulas

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$= \frac{n^2}{2} + \frac{n}{2} \quad (\text{if } n \geq 1) \leq \frac{1}{2}n^2 + \frac{1}{2} \cdot n^2$$

$$= n^2$$

Let $n_0 = 1$ and $c = 1$

(2) $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \cdots + (n-1) + n$

if $n > 1$, each $i \le n$

$\le \underbrace{n + n + n + n \cdots + n + n}_{n} = n(\underbrace{1 + 1 + 1 \cdots + 1}_{n})$

$= n^2$

Let $n_0 = 1$ and $C = 1$

$\Rightarrow \sum_{i=1}^{n} i$ is $O(n^2)$.

Ex: Give a big-O bound for $n! = n(n-1)\cdots 1$

Ex: $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

$n! = n(n-1)\cdots 1$ & if $n>1$,

$\leq \underbrace{n\cdot n\cdot n\cdots\cdot n}_{n}$

$= n^n$

So $n!$ is $O(n^n)$

Ex: What about $\log_2(n!)$?

Another trick: Use known bounds and known operations.

Just saw if $n>1$, $n! \le n^n$

Rule: If $a \le b$, then $\log_x a \le \log_x b$

$$\Rightarrow \log_2(n!) \le \log_2(n^n)$$
$$= n \log_2 n$$

So $\log_2(n!)$ is $O(n \log_2 n)$ ∎

Ex: In our (later) section on induction, we'll show $\underbrace{n \le 2^n}$ for $n \ge 1$.
What big-O does this give?

$\Rightarrow$ n is $O(2^n)$

Ex: Show that $\log_2 n = O(n)$.

strategy: use known facts.

Know $n \le 2^n$

take log of both sides

& if $n > 1$

$$\log_2 n \le \log_2(2^n)$$

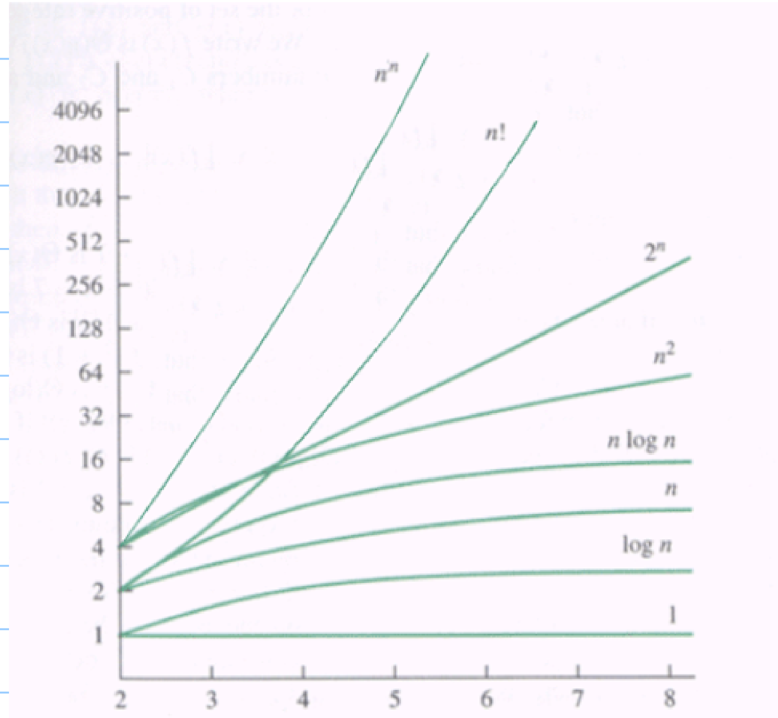
$$= n \log_2 2 = n$$

so let $C = 1$ & $n_0 = 1$, &

$\log_2 n$ is $O(n)$. $\boxed{\text{Ch}}$

Big picture

Certain functions will turn out to be very useful in the next section, which is on algorithms.



Thm:

Let $f(x)$ be a polynomial,

So $f(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

Then $f(x) = O(x^n)$.

pf: Use a fact: $|a+b| \le |a| + |b|$

$f(x) = a_0 + a_1 x + \cdots + a_n x^n \le |a_0| + |a_1| x + \cdots + |a_n| x^n$

and if $x > 1$, can multiply more x's in and only get bigger

$\le |a_0| x^n + |a_1| x \cdot x^{n-1} + |a_2| x^2 \cdot x^{n-2} + \cdots |a_n| x^n$

$= x^n \left(|a_0| + |a_1| + |a_2| + \cdots + |a_n| \right)$ let c = this $\blacksquare$

Use this theorem

Give big-O estimates for:

- $f(x) = \frac{1}{8}x^5 + x^3 + 2 = O(x^5)$
- $f(x) = 20000x^2 - 100000000x = O(x^2)$
- $f(x) = \frac{2x^2}{3000} - x + 2 = O(x^2)$

Thm: Suppose $f(x) = O(g(x))$ and $h(x) = O(p(x))$.

Then $(f+h)(x) = O(\max(g(x), p(x)))$.

pf: By definition of big-O:

$\exists\, n_0, C$ s.t. $\forall x > n_0$, $f(x) \le C \cdot g(x)$

$\exists\, n_0', C'$ s.t. $\forall x > n_0'$, $h(x) \le C' \cdot p(x)$

Consider $(f+h)(x) = f(x) + h(x)$

for any $\underbrace{x > \max\{n_0, n_0'\}}$,

$f(x) + h(x) \le C \cdot g(x) + C' \cdot p(x)$

$\le 2 \cdot \max\{C \cdot g(x), C' p(x)\}$

Let new constant $= 2 \max\{C, C'\}$ and new $n_0 =$

Cor: Suppose $f_1(x)$ and $f_2(x)$ are $O(g(x))$.
Then $(f_1+f_2)(x) = O(g(x))$.

Use this: $\underbrace{(5x^2+2x)}_{O(x^2)} + \underbrace{\left(\frac{1}{e}x^6 - ex^5 + 2\right)}_{O(x^6)}$

$= O(x^6)$

Similarly,

Thm: Suppose $f(x) = O(g(x))$ and $h(x) = O(p(x))$

Then $(f \cdot h)(x) = O(g(x)\,p(x))$

Ex: $f(x) = \underbrace{3x^2 - 5}_{= O(x^2)}$ $\qquad h(x) = \underbrace{6x \log x}_{= O(x \log x)}$

$$f(x) \cdot h(x) = (3x^2 - 5)(6x \log x)$$
$$= O(x^2 \cdot x \log x) = O(x^3 \log x)$$

Ex: Give a big-O estimate for

$$f(n) = \underbrace{3n\log(n!)} + \underbrace{(n^2+3)\log n}$$

$$(3n)\,(\log(n!)) \qquad + \qquad = O(n^2 \cdot \log n)$$

$$= O(n) \qquad = O(n\log n)$$

$$O(n^2\log n)$$

$$= O(n^2\log n)$$

Big-Omega

Dfn: Let f & g be functions from $\mathbb{R} \to \mathbb{R}$ (or $\mathbb{Z} \to \mathbb{R}$).

We say $f(x)$ is $\Omega(g(x))$ if $\exists$ positive constants C and n_0 such that

$$|f(x)| \geq C|g(x)| \quad \text{when} \quad x > n_0.$$

Read - f is big-omega of g.

Ex: Show $f(x) = x^2$ is $\Omega(x)$.

Need c & n_0.

if $x \geq 1$

$x^2 \geq x$

Let $n_0 = 1$ and $c = 1$

Thm in book: if f is $O(g)$ then g is $\Omega(f)$

Ex: Show $f(x) = 8x^3 + 5x^2 + 7$ is $\Omega(x^3)$

$$f(x) = 8x^3 + 5x^2 + 7$$
$$> 8x^3 + 5x$$
if $x > 1$,
$$> 8x^3$$

Let $n_0 = 1$ and $c = 8$

Note: Similar theorems:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = \Omega(x^n)$$

Dfn: Θ: big-theta

If f is $O(g(x))$ and $\Omega(g(x))$

$$\Rightarrow \Theta(g(x))$$