Math 135: Discrete Mathematics, Fall 2012
Homework 7

Due *in class* on Monday, Nov. 5

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1. Let f_n be the n^{th} Fibonacci number, defined as $f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0$ and $f_1 = 1$. (Hint: Remember, induction is your friend when doing recurrences!)

   (a) Prove that $\sum_{i=1}^{n}(f_i)^2 = f_n f_{n+1}$ whenever n is a positive integer.

   (b) Show that $f_{n+1}f_{n-1} - (f_n)^2 = (-1)^n$ when n is a positive integer.

2. The set of leaves and the set of internal nodes of a full binary tree can be defined recursively, using the definition of $T_1 \cdot T_2$ from class or as it is presented in section 5.3 of the book.

   - Basis step: The root r is a leaf of the full binary tree with one vertex. This tree has no internal vertices.
   - Recursive step: The set of leaves of the tree $T_1 \cdot T_2$ is the union of the sets of leaves of T_1 and T_2. The internal vertices of $T_1 \cdot T_2$ are the (new) root as well as the union of the sets of internal vertices of T_1 and T_2.

   Use induction to show that the number of leaves of a full binary tree T is one more than the number of interval vertices in a full binary tree.

3. Find a recurrence relation for the number of ways a person can climb n stairs if the person can take 1, 2, or 3 stairs at a time. (Don't forget the base cases.) Now, how many ways are there for a person to climb 8 stairs?

4. Give *exact* solutions to the following recurrences. Show your work.

   (a) $A(n) = 5A(n-1) - 4A(n-2)$, $A(0) = 0$, $A(1) = 3$.

   (b) Find the solution to the same recurrence as part (a), with $A(0) = 3$, $A(1) = 4$.

   (c) $C(n) = -5C(n-1) - 6C(n-2) + 3 \cdot (-2)^n$, $C(0) = 0$ and $C(1) = -2$.

5. Given *general form* solutions to the following recurrences. (Note: this means you don't have to solve for the constants!)

   (a) $a_n = 8a_{n-2} - 16a_{n-4} + 3n - 2$

   (b) $b_n = 8b_{n-1} - 16b_{n-2} + (n^2 - 1)4^n$

   (c) $c_n = 5c_{n-1} - 8c_{n-2} + 4c_{n-3} + n^3 - n^2$