Math 135 - Sets & Functions

Note Title 9/15/2010

Announcements

- HW1 graded
  - regrade request by 2 weeks from today
- HW3 is out, due next Wed. (23^{rd})
- HW4 will be out next Wed, due the following Wed. (30^{th})
- The first exam will be Friday the 7^{st}
  Review session on Wed. the 30^{th}

Tuples

In sets, order doesn't matter: $\{1,2\} = \{2,1\}$

also $\{1,1,2\} = \{1,2\}$

(But sometimes, order **does** matter!)

A **tuple** is an ordered list of objects

Ex:
- $(4,2,8)$
- $(\ )$
- $(\emptyset, \{2\}, \{3,8\})$
- $(1,2) \neq (2,1)$
  $(1,1,2) \neq (1,2)$

A tuple with n entries is an **n-tuple**.
(If $n=2$, called **ordered pair**.)

Cartesian Product

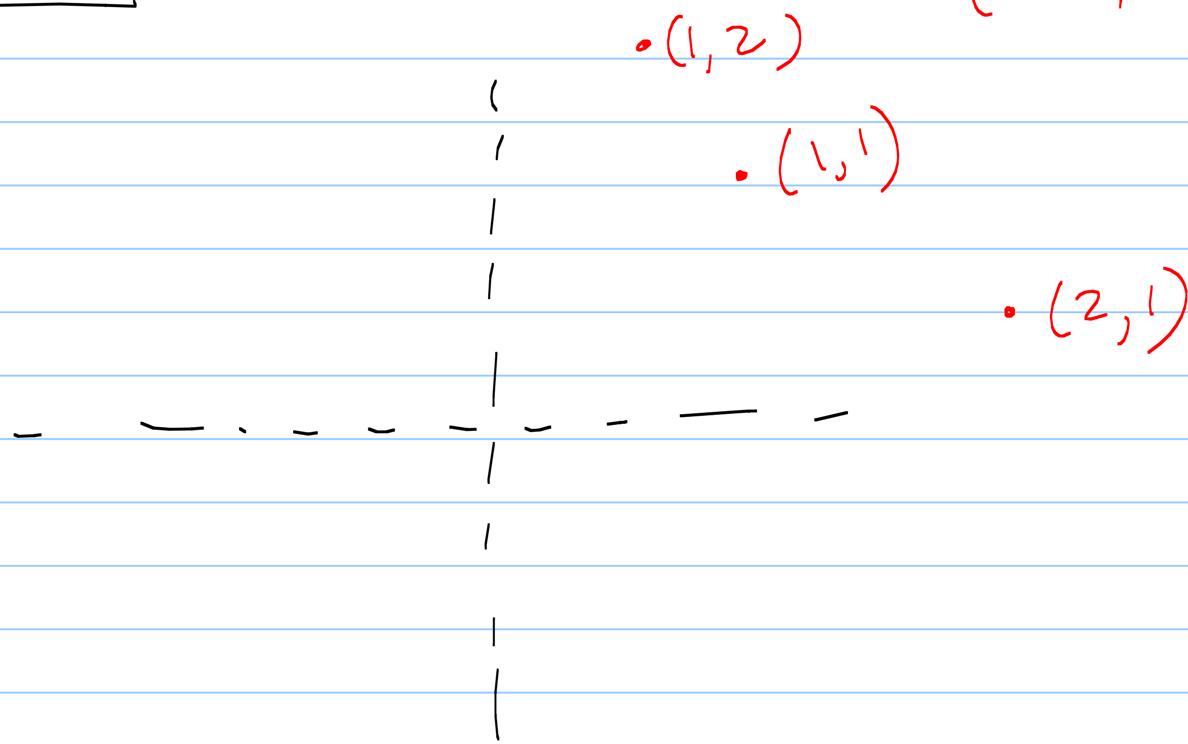
Dfn: Given sets A & B, the product of A & B (written $A \times B$) is the set of 2-tuples where first element is from A and second element is from B.

$$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$$

Ex: $A = \{a, b, c\}$ $\quad B = \{1, 2\}$

$$A \times B = \{ (a,1), (a,2), (b,1), (b,2), (c,1), (c,2) \}$$

Another: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$



With more than 2 sets, have:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, \ldots, a_n) \mid \forall i,\ a_i \in A_i\}$$

Notation: $A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$

(Hence, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, & $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$)

__Caution!__ $(A \times B) \times C \neq A \times B \times C$

Typical element in $(A\times B)\times C$: $((a,b), c)$

But in $\underline{A \times B \times C}$: (a,b,c), $a \in A$, $b \in B$, $c \in C$

__Another:__ What is $\phi \times \{a, b\}$?

$$= \{(x,y) \mid x \in \phi \text{ and } y \in \{a,b\}\}$$

$$= \phi = \{\}$$

Why?

$\phi \notin \phi$

(but $\phi \subseteq \phi$)

Instead: $\{\phi\} \times \{a, b\} = \{(\phi, a), (\phi, b)\}$

Russell's Paradox

- Sets are basic mathematical objects – but be careful of contradictions!

Ex: Let A be the set of sets which do not contain themselves:

$$A = \{ S \mid S \notin S \}$$

$$\emptyset \notin \emptyset \Rightarrow \emptyset \in A$$

Question: What about A?

Is $A \in A$?

Now every element in A is a set which does not contain itself. So $A \in A$ is impossible.

If $A \in A$, then A shouldn't be in A.

But then $A \notin A$, so A is a set which does not contain itself.

That means $A \in A$ by definition!

Solution: to keep mathematics whole, we declare that A is not a set!

Most set theory begins w/ assumption that $\emptyset$ + $\mathbb{N}$ are sets, + provides rules for building up.

Ex rule: If S is a set, $P(S)$ is a set.
2^S

These rules don't allow us to construct A.

In practice, we won't worry too much - our sets will be legal.

(See Naive Set Theory by Halmos if you're curious, or go take a logic course. :-))

Functions

$f(x) = \frac{x+2}{3}$

$f(2)$ cannot be 2 thing

Let A & B be sets. A function from A to B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ where $a \in A$, $b \in B$.

Often write $f: A \to B$ to denote a function f.

A is the domain of f, & B is the co-domain.

Examples

① $f: \overset{\text{domain}}{\mathbb{R}} \to \overset{\text{codomain}}{\mathbb{R}}$

$$\underbrace{f(x) = x + 1}$$

② Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Domain: $\{T, F\} \times \{T, F\}$

Codomain: $\{T, F\}$

(3) $f: \{1, 2, 3, 4, 5\} \to \mathbb{N}$ ← codomain

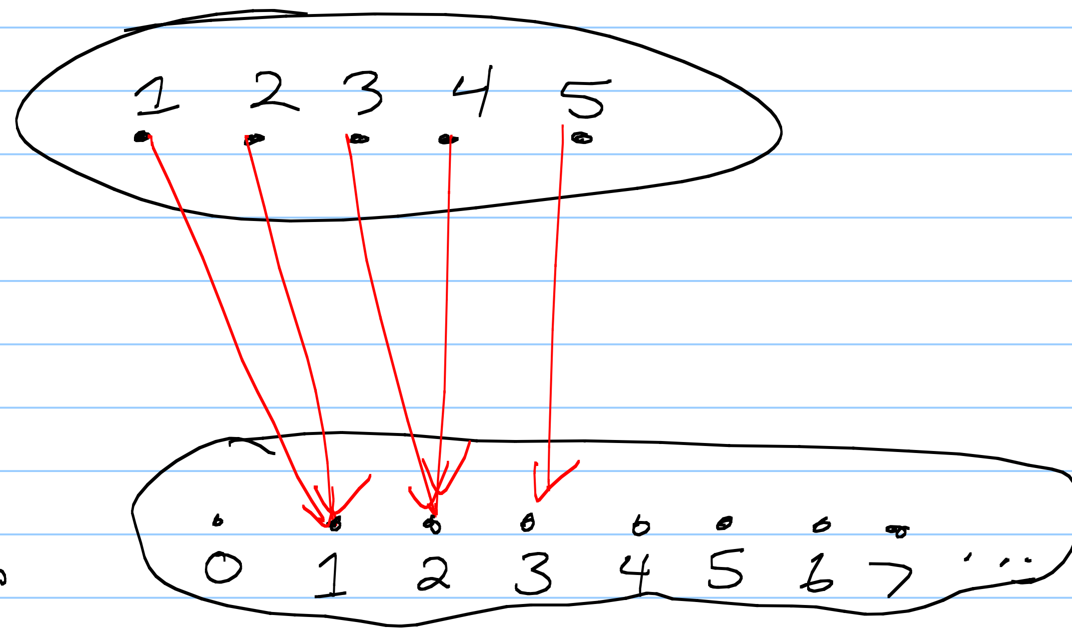
$f(x) = \left\lceil \frac{x}{2} \right\rceil$ ← ceiling function

domain

$f(1) = \left\lceil \frac{1}{2} \right\rceil = 1$

$f(2) = \left\lceil \frac{2}{2} \right\rceil = \lceil 1 \rceil = 1$

$f(3) = \left\lceil \frac{3}{2} \right\rceil = 2$



codomain

<u>Ex:</u> Let $X = \{a, b, c\}$ and $c : P(X) \to P(X)$ be the function:

$$\boxed{c(A) = X - A}$$

domain $P(X)$: $\emptyset \quad \{a\} \quad \{b\} \quad \{c\} \quad \{a,b\} \quad \{b,c\} \quad \{a,c\} \quad \{a,b,c\}$

codomain $P(X)$: $\emptyset \quad \{a\} \quad \{b\} \quad \{c\} \quad \{a,b\} \quad \{b,c\} \quad \{a,c\} \quad \{a,b,c\}$