Math 135 – More Recurrences

Note Title 10/22/2010

Announcements

- New HW posted
  - due next Friday

Recurrence Relations

- Use them to model counting problems
- Useful for runtime analysis of recursive algorithms (more next time)
- To solve:
  - unrolling ✓ (today)
  - induction ✓
  - more advanced techniques such as Master thm & characteristic eqn method (after break)

What is a recursive definition?

2 things

1) Base case(s)

Ex: $f_0 = 0$ $\qquad a_1 = 2$

$f_1 = 1$

2) Definition for a_n in terms of smaller terms $a_1 \ldots a_{n-1}$

$f_n = f_{n-1} + f_{n-2}$ $\qquad a_n = 2 \cdot a_{n-1}$

$T(n) = T(\frac{n}{2}) + 1$

Ex: Bit strings with no 2 consecutive 0's.

Ex: 1101111
101010
~~10011~~

Let b_n = # of bit strings w/ no 2 consecutive 0's of length n

$b_1 = 2 \rightarrow 0, 1$

$b_2 = 3 \rightarrow 10, 11, 01$

$b_3 = 5 \rightarrow 111, 011, 101, 110, 010$

b_{n-1}, b_{n-2}, b_{n-3} ....

Formula for b_n:

Consider the last bit: $\underbrace{- - - \ \cdots \ - \ \boxed{}}_{n \text{ bits}}$ last bit

What could it be?

__Case 1:__ 1 $\underbrace{}_{n-1}$ 1

$\underline{b_{n-1}}$ bitstring of length n-1

__Case 2:__ 0 $\underbrace{}$ 1 0

b_{n-2}

total: $\boxed{b_n = b_{n-1} + b_{n-2}}$

Recursively defined sets

Consider an inductive dfn for a set:

Base step: $3 \in S$

Recursive step: If $x \in S$ and $y \in S$, then $x+y \in S$.

let $x=3$, $y=3$

So what are some elements of S?

$$S = \{3, 6, 9, 12, 15, 18, \ldots\}$$

recursively defined set

A

Claim: $S = \{$positive integers divisible by 3$\}$

pf: How do we show 2 sets are equal??

$S \subseteq A$:

take any $s \in S$, & show it is divisible by 3 (so $s \in A$)

Take $s \in S$.

Base Case: $s = 3$ ✓

IH: Any number in S which is $< s$ is divisible by 3.

IS: $s \in S$, $s > 3$. So since $s \in S$, $\exists\ x, y \in S$, $x, y < s$, and $x + y = s$.

By IH, x & y must be div. by 3, so s is div. by 3. $\boxtimes$

"multiples of 3"

$A \subseteq S$:

$A = \{3n, n \in \mathbb{N}\}$

induction on n:

Base Case: $n=1$: $3 \cdot 1 \in S$ by Base case in recursive definition.

IH: For $k<n$, $3 \cdot k \in S$.
(Assume $3(n-1) \in S$)

IS: Consider $3 \cdot n$

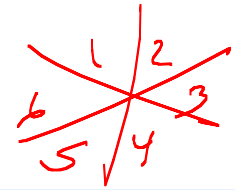
By IH, we know $3(n-1) \in S$. "x"

Also, $3 \in S$. "y"

$3(n-1) + 3 = 3n$ so $3n \in S$ $\blacksquare$

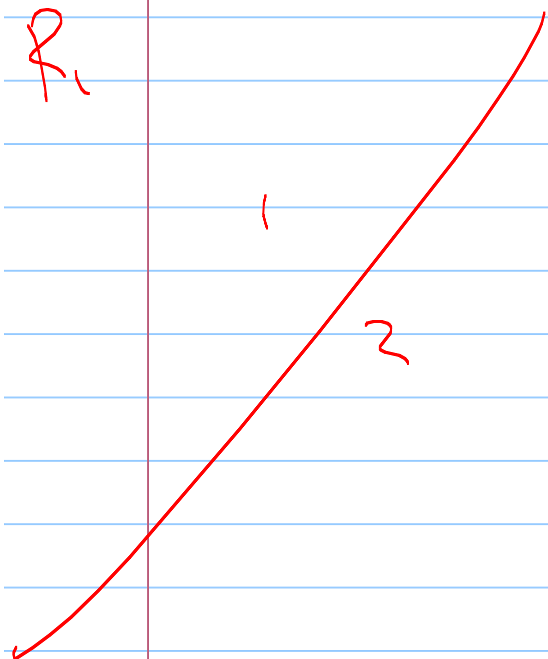
Counting regions in the plane:

(Assume no 3 lines meet at a single point)



n=1 lines

R_1



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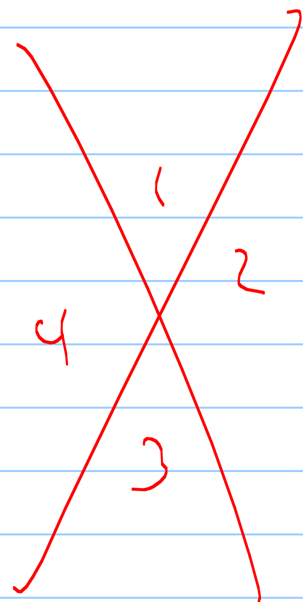
n=2 lines

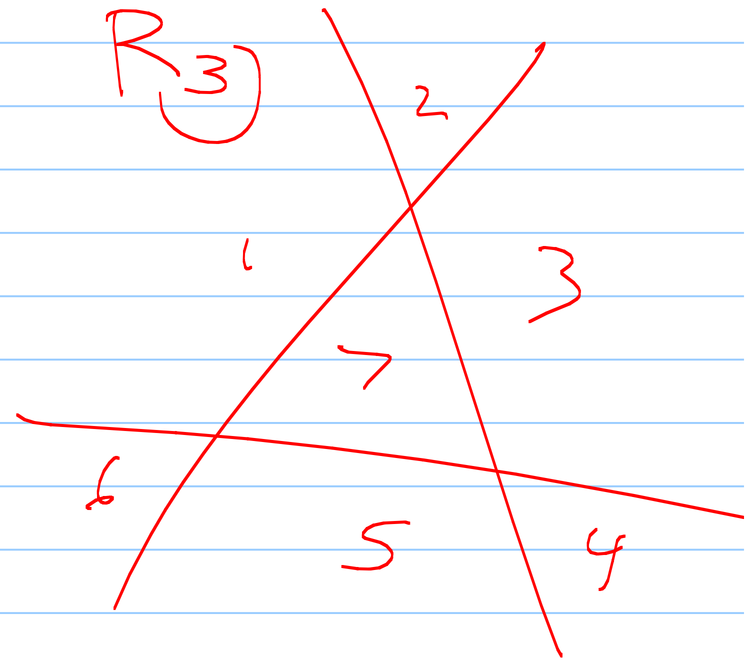
R_2



4

n=3 lines

R_3



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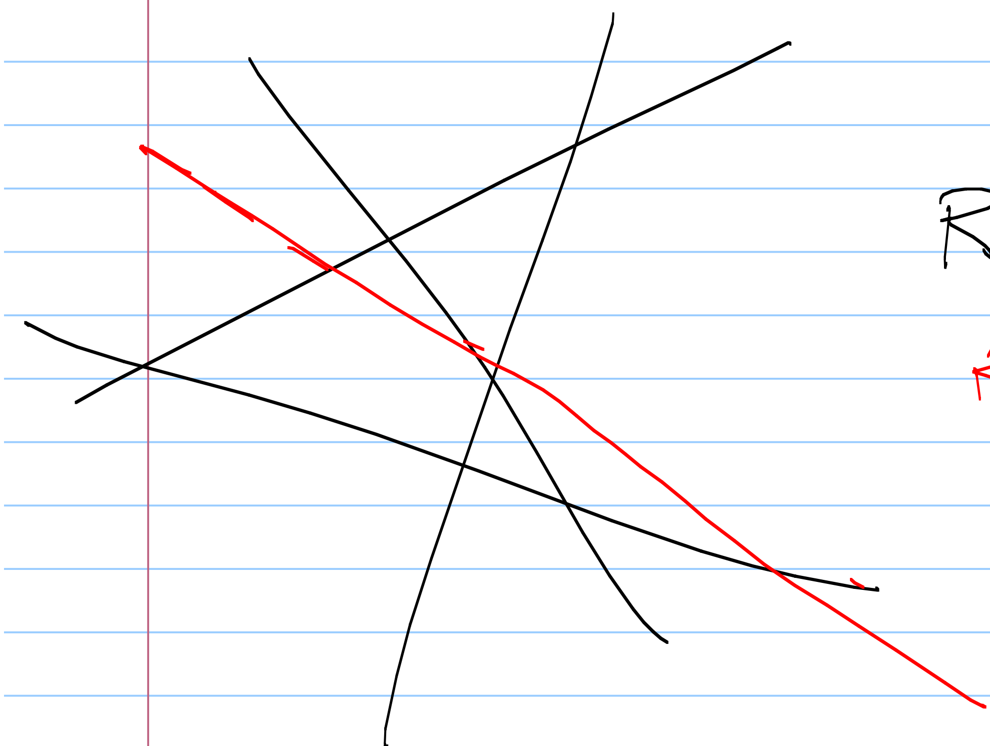
Consider $n-1$ lines w/ R_{n-1} regions.
What happens when we add an n^{th} line?
(Assume no parallel lines, so every line crosses every other line)



$$R_n = \underbrace{R_{n-1}} + n$$

$$R_3 = R_2 + 3$$

new line intersects $n-1$ other lines
divides a region into 2 regions between intersections