Math 135 - Recursion Tress

Note Title 3/22/2010

Announcements

- HW due ~~Friday~~ Monday -
- Office hours tomorrow - $\boxed{\begin{array}{l} 9\text{-}10\text{am} \\ 1\text{-}2\text{pm} \end{array}}$

Recurrences that aren't linear

$$\begin{cases} T(n) = 2T(\frac{n}{2}) + n \\ S(n) = S(\frac{n}{2}) + 1 \end{cases}$$

$$T(k) = 2T\left(\frac{k}{2}\right) + k$$

$T(1) = 1$

$$T(n) = 2\underbrace{T(n/2)}\, + n$$

Unrolling:

$$= 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

$$= 2^2 T\left(\frac{n}{4}\right) + 2\cdot\frac{n}{2} + n$$

$$= 2^2 \underbrace{T(n/4)} + 2n$$

$$= 2^2\left(2T\left(\frac{n}{8}\right) + n/4\right) + 2n$$

$$= 2^3 \underbrace{T\left(\frac{n}{8}\right)} + 3n$$

$$= 2^3\left(2T\left(\frac{n}{16}\right) + \frac{n}{8}\right) + 3n$$

$$= 2^4 T\left(\frac{n}{16}\right) + 4n$$

$\vdots$

$$= 2^i T\left(\frac{n}{2^i}\right) + i\cdot n \quad \leftarrow$$

$$T(n) = 2^i \, T\left(\frac{n}{2^i}\right) + i \cdot n$$

$$= \underbrace{d}\, T(1) + d \cdot n \qquad \frac{n}{2^d} = 1$$

$$= \underbrace{\log_2 n \cdot 1 + (\log_2 n)\, n} \qquad n = 2^d$$

$$\rightarrow \boxed{d = \log_2 n}$$

$$= O(n \log n)$$

New idea — recursion tree:

$$T(n) = 2T\left(\tfrac{n}{2}\right) + n$$

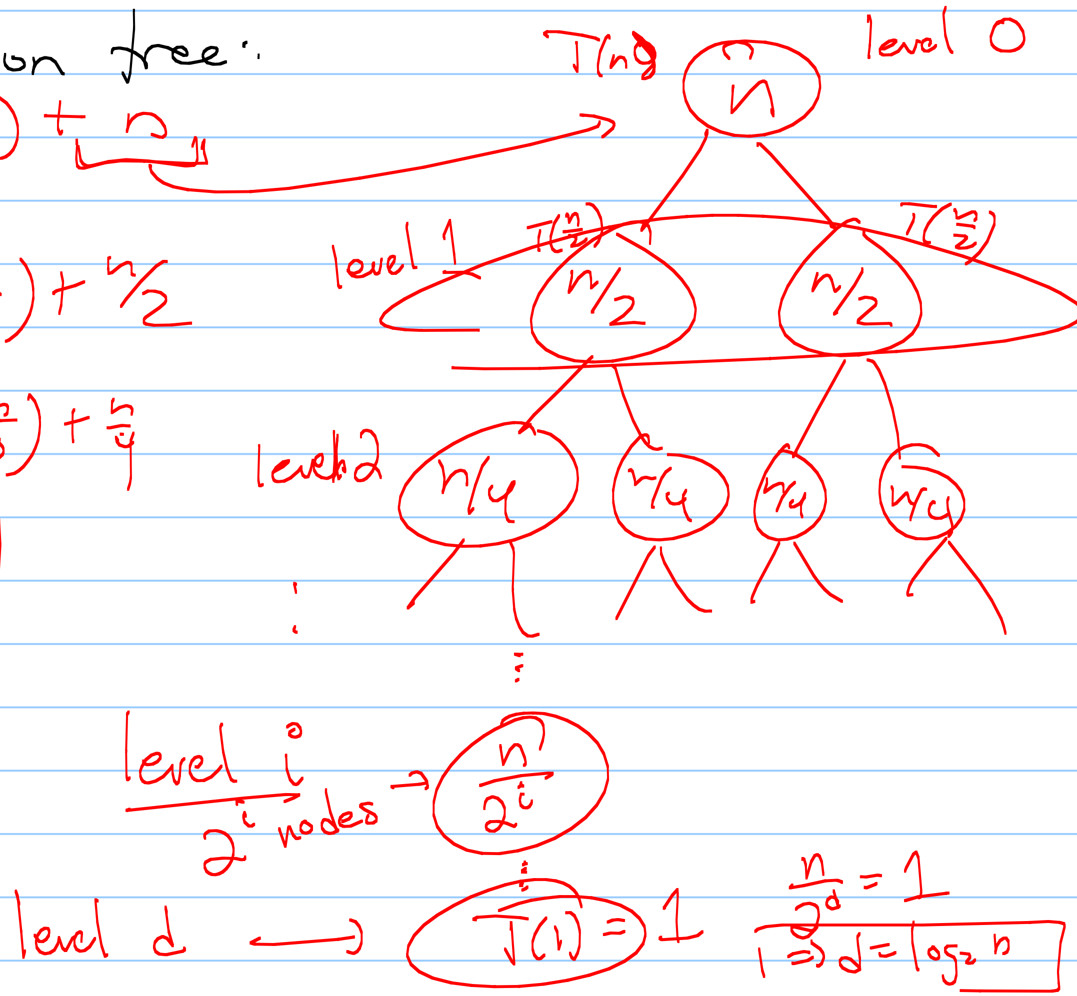
$$T\left(\tfrac{n}{2}\right) = 2T\left(\tfrac{n}{4}\right) + \tfrac{n}{2}$$

$$T\left(\tfrac{n}{4}\right) = 2T\left(\tfrac{n}{8}\right) + \tfrac{n}{4}$$

$$T(n) = \sum_{i=0}^{d} \cancel{2^i} \cdot \left(\frac{n}{\cancel{2^i}}\right)$$

$$= \sum_{i=0}^{\cancel{d}\ \log_2 n} n$$

$$= n(\log_2 n + 1)$$



$$T(k) = T\left(\frac{k}{2}\right) + 1$$

$$T(n) = 1 \cdot T\left(\frac{n}{2}\right) + 1$$

\# of children

$$T\left(\frac{n}{2}\right) = T\left(\frac{n}{4}\right) + 1$$

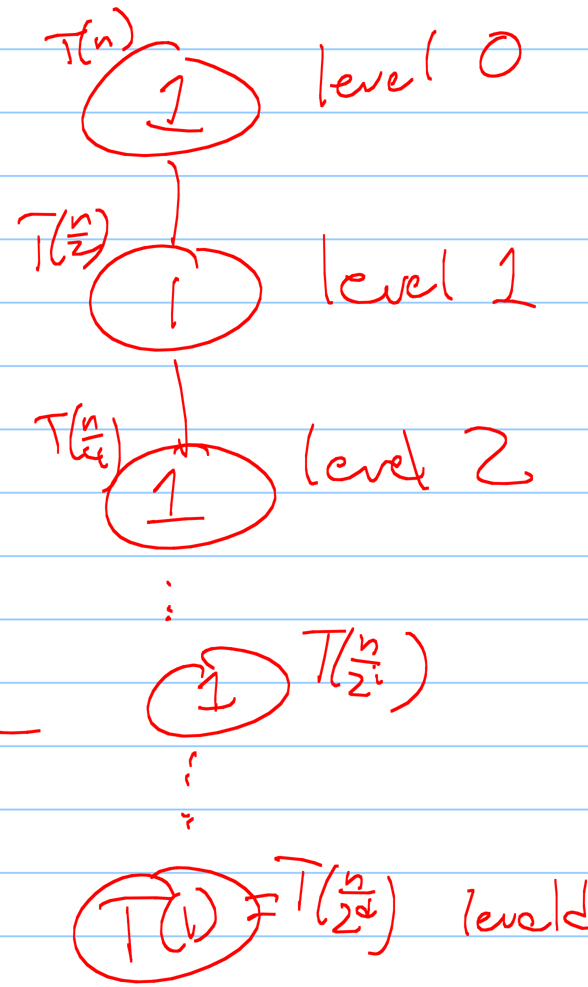
$$\frac{n}{2^d} = 1$$

$$\Rightarrow d = \log_2 n$$

$$T(n) = \sum_{i=0}^{d} 1 \cdot 1$$

$$= \sum_{i=0}^{\log_2 n} 1 = \log_2 n + 1$$

ith level: 1 node



$$S(k) = 3S\left(\frac{k}{2}\right) + k^2$$

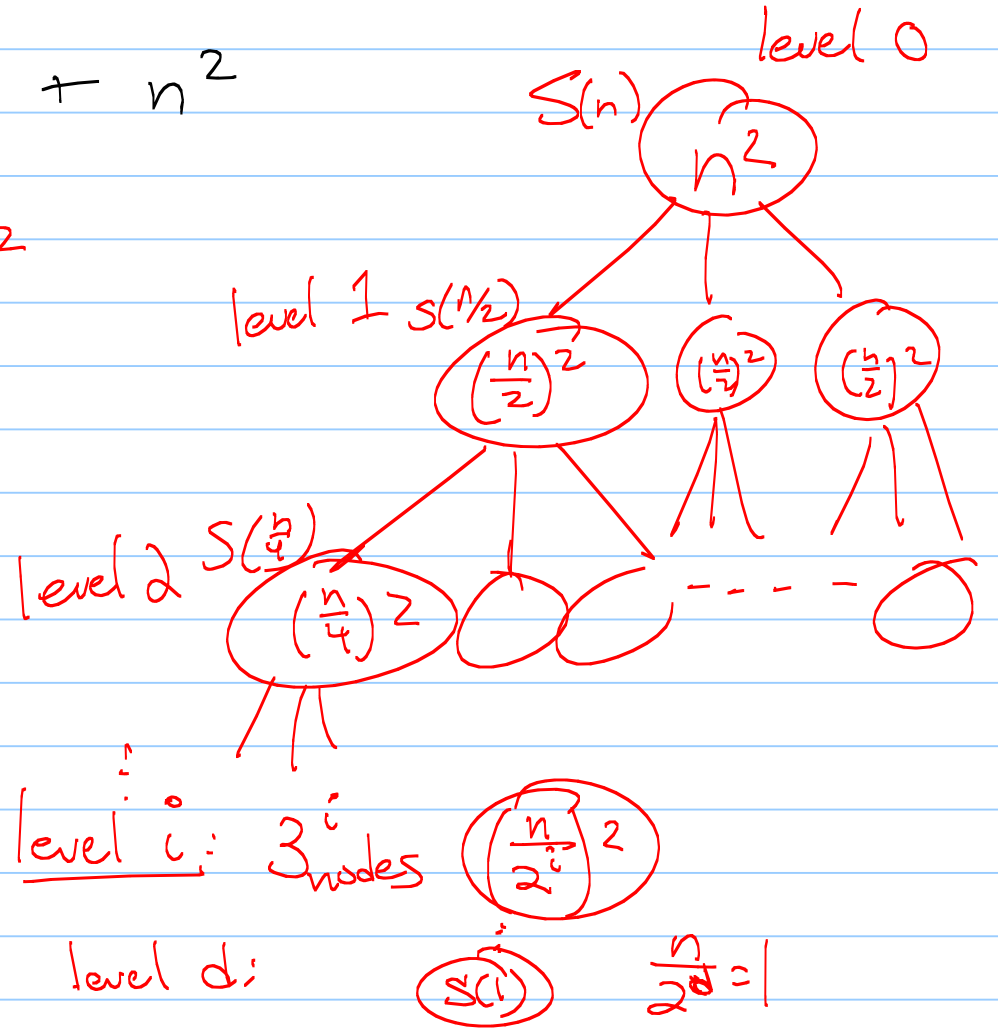
Another: $S(n) = 3S\left(\frac{n}{2}\right) + n^2$

$$S\left(\frac{n}{2}\right) = 3S\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2$$

$$S\left(\frac{n}{4}\right) = 3S\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2$$

$$S(n) = \sum_{i=0}^{d} (\#\text{ nodes})(\text{amount in each node})$$

$$= \sum_{i=0}^{\log_2 n} (3^i)\left(\frac{n}{2^i}\right)^2$$



$$\frac{a^{n+1}-a}{1-a}$$

$$\underbrace{(2^a)^b} = 2^{ba} = 2^{ab} = \underbrace{(2^b)^a}$$

$$S(n) = \sum_{i=0}^{\log_2 n} (3^i)\left(\frac{n}{2^i}\right)^2 = \sum_{i=0}^{\log_2 n} \frac{n^2 3^i}{(2^i)^2}$$

$$= n^2 \sum_{i=0}^{\log_2 n} \frac{3^i}{(2^i)^2} = n^2 \sum_{i=0}^{\log_2 n} \frac{3^i}{4^i}$$

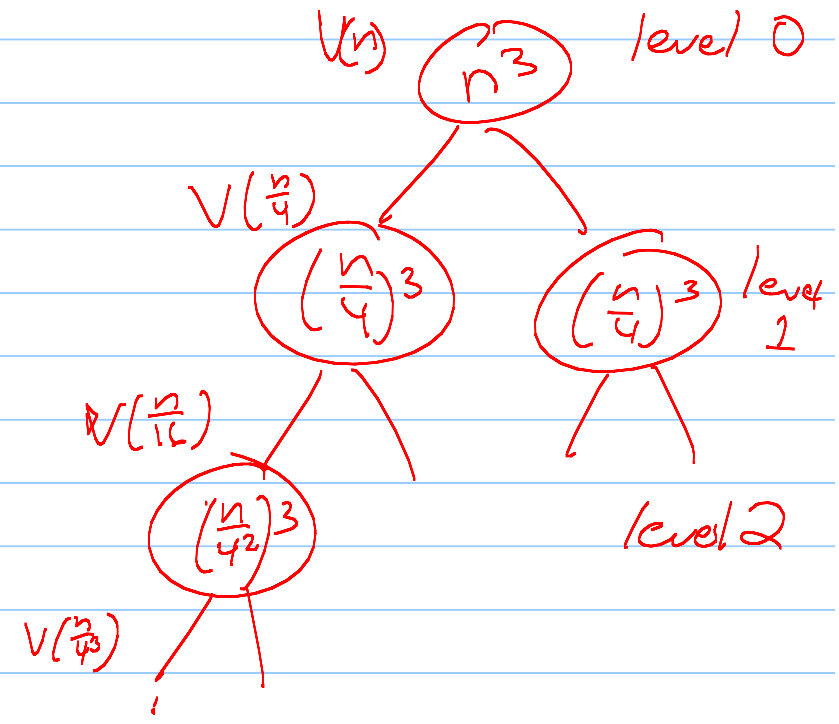
$$= n^2 \sum_{i=0}^{\log_2 n} \left(\frac{3}{4}\right)^i \qquad \text{(sec 2.4)}$$

$$\approx n^2 \left(\frac{3}{4}\right)^{\log_2 n+1} \qquad O(n^2)$$

$$V(n) = 2V\left(\frac{n}{4}\right) + \underline{n^3}$$

$$V\left(\frac{n}{4}\right) = 2V\left(\frac{n}{16}\right) + \left(\frac{n}{4}\right)^3$$

$$V\left(\frac{n}{16}\right) = 2V\left(\frac{n}{4^3}\right) + \left(\frac{n}{4^2}\right)^3$$



$$\sum_{i=0}^{\log_4 n} 2^i \left(\frac{n}{4^i}\right)^3$$

$$= n^3 \sum_{i=0}^{\log_4 n} \underbrace{\frac{2^i}{64^i}}$$

level i: 2^i $\left(\frac{n}{4^i}\right)^3$

$$\frac{n}{4^d} = 1 \Rightarrow d = \log_4 n$$

level d: $T(1)$

Next time: There is a pattern here!

We'll talk about Master theorem:

Let f satisfy $f(n) = a f(\frac{n}{b}) + \Theta(n^d)$,
where $a \geq 1$, b is an integer ≥ 1, and
c and d are real numbers, $c > 0$ & $d \geq 0$.

Then:

$$f(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$