Math 135 - Proofs

Note Title 8/27/2010

Announcements

- HW1 due Friday
- Solutions (to add #5) in text versus in separate book (can order online)

Last Time: Proofs

What is a proof?

A statement which can be rigorously shown to be true.

Direct proofs:

Think about statement $p \to q$.
When is it false?

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$\underbrace{\text{False when } p \text{ is true and } q \text{ is false.}}$

But first...

Dfn: n is an even number
$\iff n = 2k$ for some $k \in \mathbb{Z}$

n is an odd number
$\iff n = 2k+1$ for some $k \in \mathbb{Z}$

Lemma 1:

Ex: If $\overbrace{n \text{ is an odd integer}}^{p}$, then $\underbrace{n^2 \text{ is an odd integer}}_{q}$.

pf: (Assume p is true, then show q cannot be false.)

Assume n is odd.

$n = 2k+1$, where k is an integer.

$$n^2 = (2k+1)^2 = 4k^2+4k+1$$
$$= 2\underbrace{(2k^2+2k)}_{\text{is an integer,}} + 1$$

since k is an integer

so n^2 is odd $\boxed{}$

Indirect Proofs:

Recall: $p \to q$ is logically equivalent to $\neg q \to \neg p$.

(What is $\neg q \to \neg p$ called?) contrapositive

Since they are equivalent, showing $p \to q$ is true can instead be accomplished by showing $\neg q \to \neg p$.

$\underline{\text{Ex}}$: If $\overbrace{\underbrace{3n+2}_{} \text{ is odd}}^{p}$, then $\overbrace{n \text{ is odd}}^{q}$.

$\underline{\text{pf}}$: (Assume $\neg q$ is true & show $\neg p$ must also be true.)

Assume n is even.

$n = 2k$ for $k \in \mathbb{Z}$

$$3n+2 = 3(2k)+2 = 6k+2$$
$$= 2\underbrace{(3k+1)}_{\text{is an integer}}$$

So $3n+2$ is even $\blacksquare$

Ex: Prove that if $\overbrace{n = ab}^{p}$ for a, b positive integers, then $\underbrace{a \le \sqrt{n} \text{ or } b \le \sqrt{n}}_{q}$.

pf: try contrapositive ($\neg q \to \neg p$)

not ($a \le \sqrt{n}$ or $b \le \sqrt{n}$)

Assume $a > \sqrt{n}$ and $b > \sqrt{n}$.

Then $ab > \underbrace{\sqrt{n} \cdot \sqrt{n}}_{= n}$ (since $a > \sqrt{n}$ and $b > \sqrt{n}$).

So $ab \ne n$.

$\boxed{\text{Ⓜ}}$

Thm: If $\overbrace{x \text{ is even and } y \text{ is odd}}^{P}$, then $\underbrace{x+y \text{ is odd}}_{Q}$.

pf:

Assume $x = 2k$ for $k \in \mathbb{Z}$
$y = 2l+1$ for $l \in \mathbb{Z}$.

$$x+y = (2k) + (2l+1) \leftarrow \text{odd}$$
$$= 2k + 2l + 1$$
$$= 2\underbrace{(k+l)}_{\text{integer}} + 1$$

So $x+y$ is odd. ∎

Proof by cases

Thm: For every integer n, n^2+n is even.

pf: (1) n is even

$n = 2k, \quad k \in \mathbb{Z}$

$n^2 = (2k)^2 = 4k^2$

So $n^2+n = 4k^2+2k = 2\underbrace{(2k^2+k)}_{\text{integer}}$

So n^2+n is even.

(2) n is odd

by lemma 1, n^2 is odd also

$n^2+n = (2l+1) + (2k+1)$

$= 2l+2k+2 = 2\underbrace{(l+k+1)}_{\text{integer}}$

So n^2+n is even. $\blacksquare$

$\mathbb{Q}$

Dfn: A real number r is rational if $\exists\, p, q \in \mathbb{Z}$ with $q \neq 0$ such that $r = p/q$.

A real number that is not rational is called irrational: π, $\sqrt{2}$, e ...

Dfn: Reduced form:

A rational number is reduced if p & q have no common divisors.

$\frac{3}{2}$ $\quad$ $\cancel{\frac{4}{2}}$

Exercise: Prove that the sum of 2 rational numbers is rational.

(How to rewrite as $p \to q$?)

If a and b are rational, then $a+b$ is rational.

Pf: Assume $a = \frac{p_1}{q_1}$ and $b = \frac{p_2}{q_2}$

where $p_1, p_2, q_1, q_2 \in \mathbb{Z}$ and $q_1 \neq 0$ and $q_2 \neq 0$.

$$a+b = \frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$$

& $p_1 q_2 + p_2 q_1 \in \mathbb{Z}$, $q_1 q_2 \in \mathbb{Z}$, and $q_1 q_2 \neq 0$.

So $a+b$ is rational. $\boxed{m}$

<u>Proof by contradiction</u>

A <u>contradiction</u> is a logical statement which is always false.

<u>Ex</u>: $x = x + 1$

Proof by contradiction

Suppose we can show

If contradiction, then $\neg p$ must also be false. Why?

p

$\neg p$	q	$\neg p \to q$
T	T	
T	F	F
F	T	
F		

So if we want to show p is true, one method is:

- assume p is false
- derive a contradiction

(then p must be true)

Called proof by contradiction.

Prove that $\overbrace{\sqrt{2} \text{ is irrational}}^{P}$.

pf by contradiction:

Assume $\sqrt{2}$ is rational.

$$\sqrt{2} = \frac{p}{q} \qquad p, q \in \mathbb{Z} \text{ and } q \neq 0$$

(assume this is in reduced form, so p & q $\underbrace{\text{have no common factors}}$)

square: $$(\sqrt{2})^2 = \left(\frac{p}{q}\right)^2$$

$$\Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow \underbrace{2q^2 = p^2}$$

$2q^2 = p^2$ $\qquad$ p^2 is even

$\Rightarrow$ by Lemma 1 (contrapositive) p is even

$p = 2k, \quad k \in \mathbb{Z}$

$2q^2 = p^2 = (2k)^2 = 4k^2$

$\Rightarrow q^2 = 2k^2$

q^2 is even

So both p^2 & q^2 are even

$\Rightarrow$ both p & q are even

So contradiction $\lightning$ $\qquad$ $\boxtimes$