Math 135 - More induction

Note Title 9/2/2010

Announcements

- HW2 is due next Monday (may work in pairs)
- Pre-lecture slides are on website

Induction

A proof technique that is used to prove propositions of the form:

$$\underbrace{\forall n}, \; P(n)$$

$\forall n \geq 1$, $\forall n > 4$, $\forall n \geq c$

Idea:

1. Show $P(1)$ true

2. Show $\forall k > 1$, $P(k-1) \rightarrow P(k)$

Since $P(1)$ is true (by 1):

$P(1) \rightarrow P(2)$ (by 2)

$P(2) \rightarrow P(3)$ (by 2)

$P(3) \rightarrow P(4)$ (by 3)

$\vdots$

How to write inductive proofs

3 required parts

Base Case: Show $P(1)$ is true

Inductive Hypothesis: Assume $P(k-1)$

Inductive Step: Use IH to argue that $P(k)$ is true.

From worksheet:

$$\forall n \geq 1, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

pf:

Base Case: $n = 1$

LHS: $\sum_{i=1}^{1} i^2 = 1^2 = 1$

RHS: $\frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = 1$

Ind Hyp: $\sum_{i=1}^{n-1} i^2 = \frac{(n-1)(n)(2(n-1)+1)}{6} = \frac{(n-1)(n)(2n-1)}{6}$

Ind. Step: $\sum_{i=1}^{n} i^2 = \left[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2\right] + n^2 = \sum_{i=1}^{n-1} i^2 + n^2$

(apply IH) $= \frac{(n-1)(n)(2n-1)}{6} + \frac{6n^2}{6} = \frac{n}{6}\left((n-1)(2n-1) + 6n\right)$

$$= \frac{n}{6}\left((n-1)(2n-1)+6n\right) = \frac{n}{6}\left(2n^2-3n+1+6n\right)$$

$$= \frac{n}{6}\left(2n^2+3n+1\right)$$

$$= \underline{\frac{n}{6}(2n+1)(n+1)}$$

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Thm: $\forall n \geq 4$, $2^n < n! = n(n-1)(n-2)\cdots 2\cdot 1$

pf: Base case: $n=4$ $\quad 2^4 = 16$

$4! = 24$ ✓

$16 < 24$

IH: Assume $2^{n-1} < (n-1)!$

IS: $2^n = 2^1 \cdot 2^{n-1} < 2\cdot(n-1)! < n\cdot(n-1)!$

since $2 < n$

so $2^n < n(n-1)! = n!$

∎

Geometric Series:

$\forall n \geq 0,\ \displaystyle\sum_{i=0}^{n} a \cdot r^i = \frac{a \cdot r^{n+1} - a}{r-1}$

pf:

Base case: $n=0$ $\displaystyle\sum_{i=0}^{0} a \cdot r^i = a \cdot r^0 = a$ ✓

$$\frac{a \cdot r^{0+1} - a}{r-1} = \frac{a \cdot r - a}{r-1} = \frac{a(r-1)}{r-1} = a$$

IH: $\displaystyle\sum_{i=0}^{n-1} a \cdot r^i = \frac{a \cdot r^{(n-1)+1} - a}{r-1} = \frac{a \cdot r^n - a}{r-1}$

IS: $\displaystyle\sum_{i=0}^{n} a \cdot r^i = \sum_{i=0}^{n-1} a \cdot r^i + a \cdot r^n \overset{\text{apply IH}}{=} \frac{a \cdot r^n - a}{r-1} + a \cdot r^n$

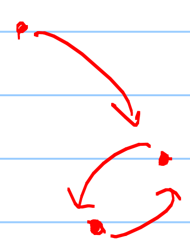
$$= \frac{a \cdot r^n - a + (r-1) a \cdot r^n}{r-1} = \frac{\cancel{a \cdot r^n} - a + a \cdot r^{n+1} - \cancel{a \cdot r^n}}{r-1}$$

Suppose n friends have a water balloon fight. Each moves to a location (so that all distances between friends are distinct).
Next, each throws their balloon at the closest target.

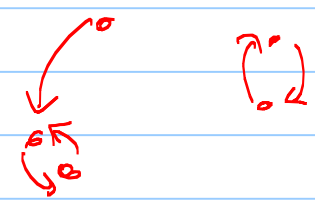
<u>Claim</u>: If n is odd, then at least one person stays dry.

prove using induction

3 people



5 people

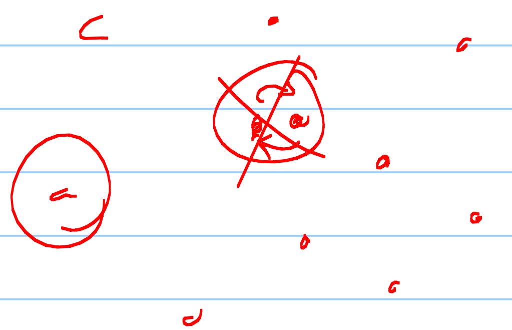


Base Case: $n=1$ He has no one to throw at him.

$n=3$: Pair that are closest throw at each other, so 3^{rd} person stays dry.

IH: Assume 1 person stays dry if we have $n-2$ in a fight.

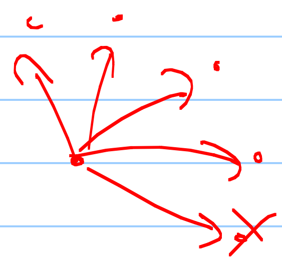
IS: Consider n people. Consider closest pair. Remove them, & consider a water fight between the rest. In this smaller fight, someone stays dry (Call him Bob.) Now add back 2 people. Neither person could throw at Bob, so Bob stays dry.



The Gossip Problem

- There are n people, each of whom knows one secret.
- Every time 2 people call each other, they tell each other *all* the secrets they know.

How many phone calls are needed for everyone to know *all* of the secrets?

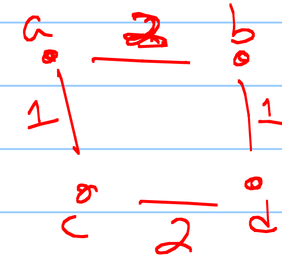


$(n-1)+(n-2) = 2n-3$

Claim: If $n \geq 4$, then $2n-4$ suffice.

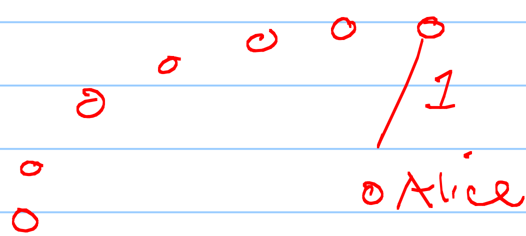
Pf: <u>Base case:</u> $n = 4$

Use $2 \cdot 4 - 4 = 4$ calls



<u>IH:</u> For $n-1$ people, $2(n-1)-4$ calls are enough.
$= 2n-6$

<u>IS:</u> n people — Alice will call 1 person.
Remove Alice, & have other $n-1$ use $2n-6$ calls.
Then have Alice call someone.



Total: $1 + (2n-6) + 1 = 2n-4$