Math 135 - More divide & conquer

Note Title 3/24/2010

Announcements

- HW due Monday
- Next HW will be out today / tomorrow
  - due 1 week from / Monday (the 15^{th}) /
- Review session on Mon the 15^{th}
- Next exam - Wed., 17^{th}

Series & Summations Recap: (2.4)

$$\sum_{k=0}^{n} a \cdot r^k = \frac{ar^{n+1} - a}{r-1} \quad (\text{if } r \neq 0)$$

Also:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad (\text{if } |x| < 1)$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = \frac{1}{1/2} = 2$$

$$T(k) = \underline{3}T\left(\frac{k}{2}\right) + k$$

So, for recursion trees:

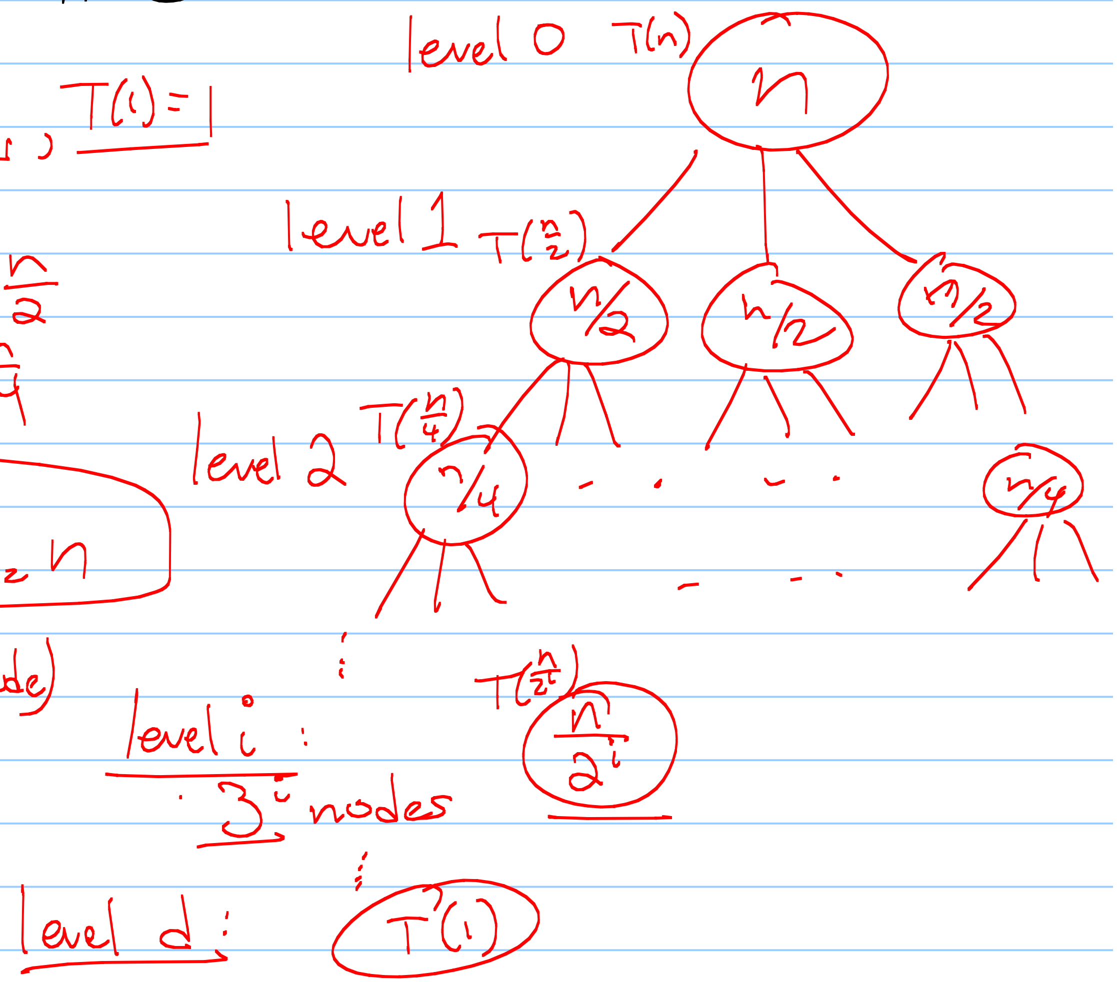
$$T(n) = 3T\left(\frac{n}{2}\right) + \underline{n}, \quad \underline{T(1) = 1}$$

$$T\left(\frac{n}{2}\right) = 3T\left(\frac{n}{4}\right) + \frac{n}{2}$$

$$T\left(\frac{n}{4}\right) = 3T\left(\frac{n}{8}\right) + \frac{n}{4}$$

$$\frac{n}{2^d} = 1 \Rightarrow \boxed{d = \log_2 n}$$

$$T(n) = \sum_{i=0}^{d} (\#\text{nodes})(\text{work per node})$$



$$T(n) = \sum_{i=0}^{d} (\#\text{nodes})(\text{work per node})$$

$$= \sum_{i=0}^{\log_2 n} (3^i)\left(\frac{n}{2^i}\right) = \sum_{i=0}^{\log_2 n} n \left(\frac{3}{2}\right)^i$$

$$= \frac{n\left(\frac{3}{2}\right)^{\log_2 n + 1} - n}{\frac{3}{2} - 1} = 2n\left(\left(\frac{3}{2}\right)^{\log_2 n + 1} - 1\right)$$

$$= 2n\left(\frac{3^{\log_2 n + 1}}{2^{\log_2 n + 1}} - 1\right)$$

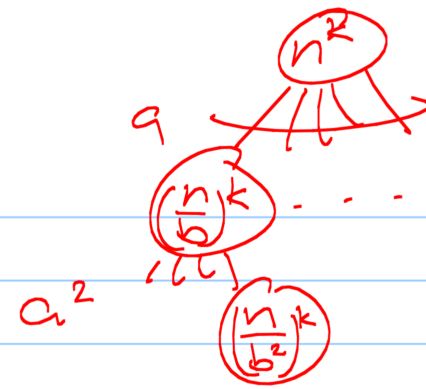
$$3^{\log_2 n + 1} = 3 \cdot 3^{\log_2 n} = 3 \cdot n^{\log_2 3}$$

$$2^{\log_2 n + 1} = 2^{\log_2 n} \cdot 2 = 2n$$

$$= \frac{3 \cdot \cancel{2n} \cdot n^{\log_2 3}}{\cancel{2n}} - 2n$$

$$= \Theta\left(n^{\log_2 3}\right)$$

Master theorem:



Let f satisfy $f(n) = a f(\frac{n}{b}) + \Theta(n^k)$, where $a \geq 1$, b is an integer ≥ 1, and c and k are real numbers, $c > 0$ & $k \geq 0$.

Then:

$$f(n) = \begin{cases} O(n^k) & \text{if } a < b^k \\ O(n^k \log_b n) & \text{if } a = b^k \\ O(n^{\log_b a}) & \text{if } a > b^k \end{cases}$$

How to use:

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n^1)$$

Here: $a = 2$
$b = 2$
$k = 1$

So: $a = 2$ $\quad b^k = 2^1 = 2$

Case 2 $\quad a = b^k$:

$$T(n) = \Theta(n \log n)$$

$$= T\left(\frac{n}{4/3}\right)$$

Ex: $T(n) = 1 \cdot T\left(\frac{3n}{4}\right) + n^{2}$

$a = 1$

$b = 4/3$

$k = 2$

So $a = 1$, $b^k = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$

$a < b$

So $T(n) = O(n^2)$

Ex: $T(n) = 3T\left(\frac{n}{2}\right) + n$

$a = 3$

$b = 2$

$k = 1$

So: $a = 3$ $\quad$ $b^k = 2^1 = 2$

$a > b^k$

So $T(n) = O\left(n^{\log_2 3}\right)$

$$T(k) = k^{1/2}\, T(k^{1/2}) + k$$

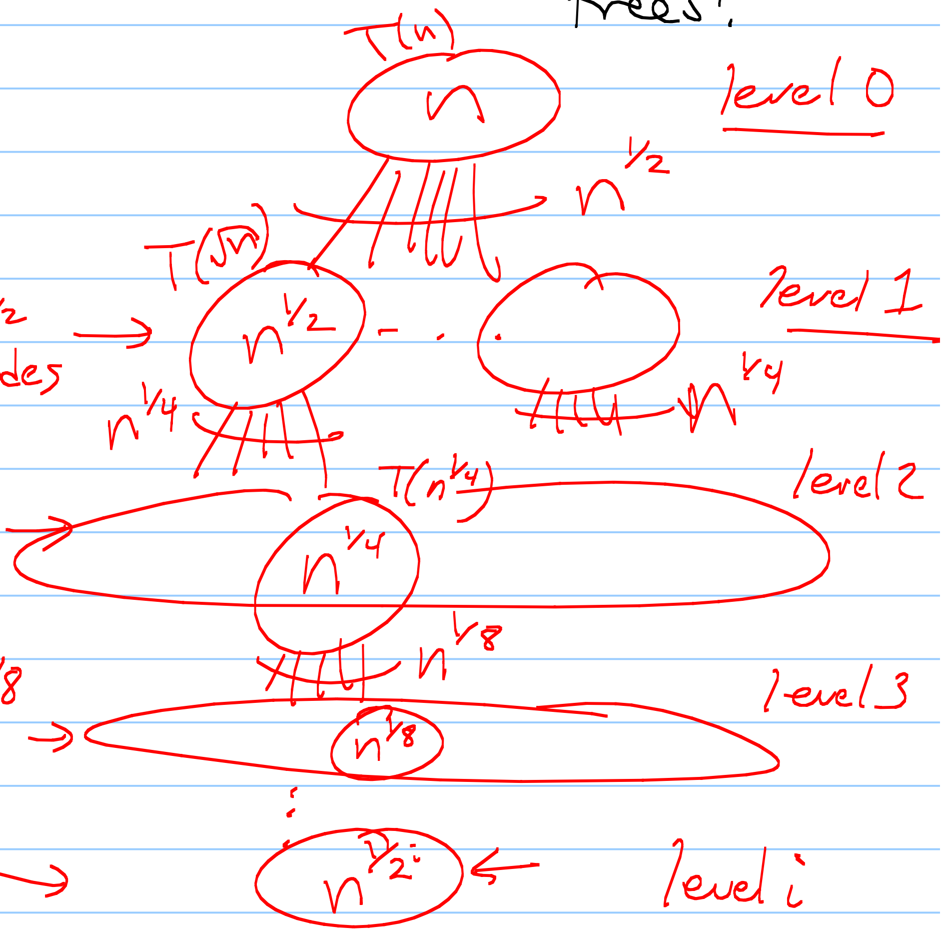
When Master thm doesn't help: use recursion trees!

$$T(n) = \sqrt{n}\, T(\sqrt{n}) + \underline{n}$$

$$T(n^{1/2}) = n^{1/4}\, T(n^{1/4}) + n^{1/2}$$

$$T(n^{1/4}) = n^{1/8}\, T(n^{1/8}) + n^{1/4}$$

$n^{1/2}$ nodes $\rightarrow$



<u>depth d</u>:

$$n^{\frac{1}{2^d}} = 2$$

$$\log_2\left(n^{\frac{1}{2^d}}\right) = \log_2 2$$

$$\frac{1}{2^d}\log_2 n = 1 \Rightarrow \log_2 n = 2^d$$

$$d = \log \log n$$

$$n^{1/2} \cdot n^{1/4} = n^{3/4} \rightarrow$$

$$n^{3/4} \cdot n^{1/8} = n^{7/8} \rightarrow$$

$$n^{1-\frac{1}{2^i}} \rightarrow$$

$$\underbrace{n + n + n \cdots + n}_{\log \log n}$$

So $T(n) = \sum_{i=0}^{d} (\text{work per node})(\#\text{nodes per level})$

$$= \sum_{i=0}^{\log_2 \log_2 n} \left(n^{1/2^i}\right)\left(n^{1-\frac{1}{2^i}}\right)$$

$$= \sum_{i=0}^{\log \log n} \left(n^{\frac{1}{2^i}+1-\frac{1}{2^i}}\right) = \sum_{i=0}^{\log \log n} n$$

$$= n \log_2 \log_2 n$$

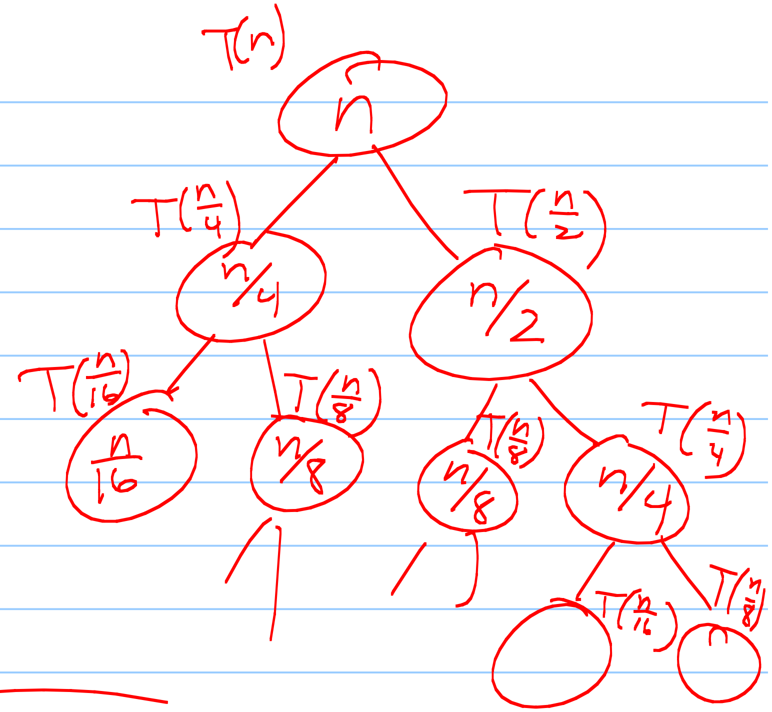
$$T(k) = T\left(\frac{k}{4}\right) + T\left(\frac{k}{2}\right) + k$$

$$T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{n}{2}\right) + n$$

$$T\left(\frac{n}{4}\right) = T\left(\frac{n}{16}\right) + T\left(\frac{n}{8}\right) + \frac{n}{4}$$

$$T\left(\frac{n}{2}\right) = T\left(\frac{n}{8}\right) + T\left(\frac{n}{4}\right) + \frac{n}{2}$$

$$T\left(\frac{n}{8}\right) = T\left(\frac{n}{32}\right) + T\left(\frac{n}{16}\right) + \frac{n}{8}$$



level i → total work: $\left(\frac{3}{4}\right)^i n$

$\log_4 n$

depth: $\log_4 n \le d \le \log_2 n$

$\log_2 n$

So:

$$T(n) \le \sum_{i=0}^{\log_2 n} \left(\tfrac{3}{4}\right)^i n \le n \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i$$

$$\le n\left(\frac{1}{1-3/4}\right) = 4n$$

$$T(n) = O(n)$$

$$T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{n}{2}\right) + n \ge n$$

So $T(n) = \Omega(n)$ $T(n) = \Theta(n)$