Math 135 - Predicates & Quantifiers

Note Title 8/26/2010

Announcements

- Turn in HW0
- HW1 is posted, due next Friday

So far:

- Propositions
- negations
- truth tables
- implications $p \to q$

Ch 1, Sections 1–3

Predicates : $P(x)$

propositions that depend on some variable

Ex: $x > 3 \longrightarrow P(x)$

$P(4)$ is true, $P(1)$ is false

$x = y + 3 \longrightarrow Q(x, y)$

$Q(4,1)$ is true

$Q(1,4)$ is false

"x is in discrete math" $\longrightarrow R(x)$

"x is a SLU student" $\longrightarrow S(x)$

Note:

- Can combine these
- Truth value depends on variable

$P(5) = \text{true}$

$Q(1, 11) = \text{false}$

$\underbrace{R(x) \wedge S(x)}$

(true if x is a student in this class)

Quantifiers

$\mathbb{N}$ - natural numbers
0, 1, 2, ...

$\mathbb{R}$ - real numbers
$\mathbb{Q}$ - rational numbers
$\mathbb{Z}$ - integers

$\underbrace{\forall x}_{\text{Universal quantifier}} P(x)$: $\underbrace{\text{For all}}$ x (in universe), $P(x)$ is true.

Ex: Let $P(x) =$ "$x+1>x$", and $Q(x) =$ "$x<2$"

What are the truth value of:

$\underbrace{\forall x \overset{\text{in}}{\in} \mathbb{R}}$, $P(x)$: True

for all x in real numbers, $x+1>x$

$\forall x \in \mathbb{R}$, $Q(x)$: False - consider 3

for all x in reals, $x<2$

Quantifiers

$\underbrace{\exists x}$ $P(x)$: There exists x (in universe) such that $P(x)$ is true.

Existential quantifier — there exists an x

Ex: Let $P(x) =$ "$x > 3$" and $Q(x) =$ "$x = x+1$"

$\exists x \in \mathbb{R}, P(x)$: True
there exist a real number x with $x > 3$

$\exists x \in \mathbb{R}, Q(x)$: False
there exists x in reals such that $x = x+1$

These can get more complicated:

$$\underbrace{\exists x \, (P(x) \wedge Q(x))} \vee \forall x \, R(x)$$

Which quantifier holds where?

There is an x $\overbrace{\text{s.t.}}^{\text{such that}}$ $P(x)$ and $Q(x)$ hold

or for all x, $R(x)$ holds.

$$\exists x \left((P(x) \wedge Q(x)) \vee R(x) \right)$$

Negations

Let the universe be all SLU students.

How should we negate quantifiers?

Consider the following:

$P(x) =$ "x has taken college algebra."

So $\forall x \, P(x)$ is?

All SLU students have taken college algebra.

What is $\neg(\forall x \, P(x))$? There is some SLU student who has not taken college algebra.

$$\neg(\forall x \, P(x)) = \underbrace{\exists x \, \neg P(x)}$$

$\cancel{\forall x \, \neg P(x)}$

What about $\exists x\, P(x)$?

There is a SLU student who has taken college algebra.

$\neg(\exists x\, P(x))$?

No SLU students have taken college algebra.

$$\neg(\exists x\, P(x)) = \forall x\, \neg P(x)$$

$$\neg(p \vee q) = \neg p \wedge \neg q$$

So negations:

$$\neg(\exists x\, P(x)) = \forall x\, \neg P(x)$$

$$\neg(\forall x\, P(x)) = \exists x\, \neg P(x)$$

These "stack":

$$\neg(\forall x\, \forall y\, P(x,y)) = \exists x\, \exists y\, \neg P(x,y)$$

$$\neg(\forall x\, (P(x) \vee Q(x))) = \exists x\, \neg(P(x) \vee Q(x))$$

$$= \exists x\, (\neg P(x) \wedge \neg Q(x))$$

Nested quantifiers:

$\forall x \overset{\in\mathbb{R}}{} \exists y \overset{\in\mathbb{R}}{} (x+y=0)$ true

translate: For all x, there is y s.t. $x+y=0$.

What about: False

$\exists y \forall x \ (x+y=0)$:

If y is fixed first, $x+y \neq 0$ for any other y.

Another one: If our universe is $\mathbb{R}$,

$$\forall x \forall y \, (((x>0) \wedge (y<0)) \rightarrow (xy<0))$$

For all x and for all y,

if $x>0$ and $y<0$

then $xy<0$.

Negating implications

What is $\neg(p \to q)$?

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$p \wedge \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

not the same as converse or inverse

Ex: Write negation of: "If Bob has an 8am class today, then it is Tuesday."

Bob has an 8am class and it is not Tuesday.

$\mathbb{R}^{\geq 0}$

So:

What is $\neg\left(\forall x\left(P(x) \to Q(x)\right)\right)$

$$\exists x\left[\neg\left(P(x) \to Q(x)\right)\right]$$

$$= \exists x\left(P(x) \wedge \neg Q(x)\right)$$

Ex: Write the negation of:

$\forall$ real numbers $x > 0$, if $x^2 = 1$, then $x^3 = 1$.

$\exists$ real number $x > 0$, $x^2 = 1$ and $x^3 \neq 1$.

$\neg(x > 3)$ $\quad x \leq 3$

Proofs:

- A theorem (or lemma, or proposition) is a statement that can be rigorously shown to be true.

- The sequence of statements giving that argument is called a proof.

Direct proofs:

Think about statement $p \to q$.
When is it false?

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

False when p is true and q is false.

Ex: If n is an odd integer, then n^2 is an odd integer.

pf: (Assume p is true, then show q cannot be false.)

Assume n is an odd integer.

Write $n = 2m+1$ for some $m \in \mathbb{Z}$.

$$n^2 = (2m+1)^2$$
$$= \underbrace{4m^2 + 4m}\ + 1$$
$$= 2(2m^2+2m)+1$$

So n^2 is equal to $2*(\text{number})+1$.
So n^2 is odd. ∎