Math 135 - Solving Recurrences (pt. 2)

Note Title 3/19/2010

Announcements

- HW due
- Next HW up tonight, due in 1 week

Characteristic Equation Method

Method for inhomogeneous recurrences:

1. "Ignore" $g(n)$ and find general solution for the rest

2. Find general solution for $g(n)$

3. Add them together

4. Use base cases (& possibly recurrence) to solve for constants.

How to do it when $g(n) =$ (polynomial of deg k) $* s^n$:

$\boxed{\text{Is } s \text{ a characteristic root?}}$

No

try a general solution of the form (polynomial of degree k) $\cdot s^n$

Note: use different constants

Yes

what is its multiplicity? Let this be m.

try a general solution of the form n^m(poly. of degree k) $\cdot s^n$

Ex: $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

don't solve for constants

(just do general form)

① $a_n = 5a_{n-1} - 6a_{n-2}$

poly: $x^2 = 5x - 6 \Rightarrow x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

roots $= 2, 3$

so $a_n = \underbrace{c_1 \cdot 2^n + c_2 \cdot 3^n}$

② $g(n) = (1) \cdot 7^n$ $\quad s = 7$ $\quad$ poly deg $= 0$

polynomial

7 is not a root

so $\underbrace{c_3 \cdot 7^n}$

3 $\boxed{a_n = c_1 2^n + c_2 3^n + c_3 7^n}$

general form

base cases

$f_0 = \cancel{0} = \cancel{2} \quad 0$

$f_1 = \cancel{1} = \cancel{3} \quad 0$

$f_n = f_{n-1} + f_{n-2}$

Ex: $a_n = 6a_{n-1} - 9a_{n-2} + n \cdot 3^n$

① polynomial: $x^2 = 6x - 9 \Rightarrow x^2 - 6x + 9 = 0$

$(x-3)(x-3) = 0$

root: 3, with multiplicity 2

$c_1 3^n + c_2 \cdot n \cdot 3^n$

② $g(n) = n^1 \cdot 3^n$

deg of poly = 1

$s = 3$ ← char root!

so: $n^2(c_3 n + c_4) \cdot 3^n$

③ $\boxed{a_n = c_1 3^n + c_2 n 3^n + n^2(c_3 n + c_4) 3^n}$

Ex: $a_n = 6a_{n-1} - 9a_{n-2} + n^2 2^n$

① $x^2 = 6x - 9$: $c_1 3^n + c_2 n 3^n$

roots: 3, w/mult. 2

② $g(n) = n^2 \cdot 2^n$

deg of poly: 2 $s = 2$ ← not a char root

try: $(c_3 n^2 + c_4 n + c_5) \cdot 2^n$

③ $a_n = c_1 3^n + c_2 n 3^n + (c_3 n^2 + c_4 n + c_5) 2^n$

Ex: $a_n = 6a_{n-1} - 9a_{n-2} + (n^2+1)3^n$

(1) $\rightarrow c_1 3^n + c_2 n 3^n$

(2) $g(n) = (n^2+1)3^n$

degree $= 2$ $\quad \underbrace{s = 3}_{\text{is a root}}$

mult $= 2$

$\Rightarrow n^2 [c_3 n^2 + c_4 n + c_5] \cdot 3^n$

n

Ex: $a_n = 1a_{n-1} + n$, $a_1 = 1$, $a_0 = 0$

Another way to solve — recursion!

1) equation: $x = 1 \leftarrow$ root $\quad \underbrace{c_1 \cdot 1^n}$

2) $g(n) = \underbrace{n}_{\text{deg of poly} = 1} \cdot 1^n \quad \underbrace{s = 1}$

form: $\underbrace{n^1 (c_2 n + c_3) 1^n}$

3) $a_n = c_1 \cdot 1^n + n(c_2 n + c_3) \cdot 1^n$

$$a_0 = 0$$

$$a_n = c_1 \cdot \cancel{1^n} + n(c_2 n + c_3) \cdot \cancel{1^n}, \quad a_1 = 1$$

$$= c_1 + n \cdot c_2 \cdot n + n \cdot c_3$$

$$= c_1 + \underline{n^2} \cdot c_2 + n \cdot c_3$$

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$$a_1 = 1 = c_1 + c_2 + c_3$$

$$\boxed{a_0 = 0 = c_1}$$

$$a_2 = 3 = c_1 + 2^2 c_2 + 2c_3$$

$$\begin{cases} 1 = c_2 + c_3 & ; \quad c_2 = 1 - c_3 \\ 3 = 4c_2 + 2c_3 & \Rightarrow 3 = 4(1 - c_3) + 2c_3 \\ c_2 = \frac{1}{2} & \quad 3 = 4 - 4c_3 + 2c_3 \end{cases}$$

$$-1 = -2c_3 \Rightarrow c_3 = \frac{1}{2}$$

$$a_n = \frac{1}{2} n^2 + \frac{n}{2} = \frac{n(n+1)}{2}$$

Divide + Conquer Recurrences

(Section 7.3 or lecture notes on web)

Sometimes, we have recurrences that don't give nice characteristic equations:

$$T(n) = T(\tfrac{n}{2}) + 1$$

$$T(n) = 2T(\tfrac{n}{2}) + n$$

Unrolling:

Unrolling: