Math 135 - Solving recurrences - 7.2

Note Title 3/15/2010

Announcements

- HW due Friday
- Next HW out by Friday, due in 1 week

Solving Linear recurrences

Dfn: A linear homogenous recurrence has the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d}$$

where $c_1, \dots, c_d$ are constants and $c_d \neq 0$.

The order of the recurrence is d.

Ex: $a_n = 3a_{n-1} + 2a_{n-2}$

order 2

Examples Yes or no?

$\rightarrow P(n) = 1.06\, P(n-1)$ Yes, order 1

$H(n) = 2H(n-1) + 1$ No

$\rightarrow R(n) = R(n-1) + (R(n-2))^2$ No

$A(n) = 3A(n-5) \leftarrow$ Yes, order 5

$C(n) = n\,C(n-1) \leftarrow$ No

$F(n) = F(n-1) + F(n-2) \leftarrow$ Yes order 2

$f_n = f_{n-1} + f_{n-2}$

$$\frac{r^a}{r^b} = r^{a-b}$$

Basic Approach

Look for solutions of the form $a_n = r^n$, where r is a constant.

So if $a_n = r^n$ is a solution, have:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$$

divide by r^{n-k}

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k$$

Characteristic polynomial

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$$

So:

The sequence $a_n = \{r^n\}$ is a solution

$\iff$

r is a solution of characteristic equation $r^k - c_1 r^{k-1} - \cdots - c_k = 0$

Dfn:
The roots of the characteristic equation are called the characteristic roots

$$\underset{\substack{\uparrow\\ x}}{P(n)} = \underset{\substack{\uparrow\\ 1.06}}{1.06}\, P(n-1)$$

Ex: $P(n) = 1.06\, P(n-1)$ degree: 1

Char eqn: $(x - 1.06) = 0$

Root: $x = 1.06$

constant = 1

Ex: $F(n) = 1F(n-1) + 1F(n-2)$

order 2

Char eqn:

$$x^2 = 1 \cdot x + 1$$

$$x^2 - x - 1 = 0$$

$a = 1,\ b = -1,\ c = -1$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

2 characteristic roots: $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$

Ex: $A(n) = A(n-1) + 2A(n-2)$ degree 2

poly: $x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$ roots: $2, -1$

Ex: $B(n) = 2B(n-1) - B(n-2)$

$x^2 = 2x - 1$

$x^2 - 2x + 1 = 0$ root: 1 with multiplicity 2

$(x-1)(x-1) = 0$

Ex:

$$a_n = 2a_{n-2} + a_{n-3}$$

$${}_{n-1}$$

$$\rightarrow x^3 = 2x + 1$$

Finding General Solutions

- If r is a non-repeated root of the characteristic equation, then r^n is a solution to the recurrence.
- If r is a repeated root with multiplicity k, then
$$r^n, \; n \cdot r^n, \; \ldots, \; n^{k-1} \cdot r^n$$
are all solutions.
- Use linear combinations of these

Ex: $P(n) = 1.06\,P(n-1)$, $P(0) = 10{,}000$

$\rightarrow$ $x - 1.06 = 0$ had root $x = 1.06$

$\boxed{P(n) = c \cdot (1.06)^n}$ (c is constant)

use base case to solve for c:

$$P(0) = 10{,}000 = c\,(1.06)^0$$

$$\Rightarrow c = 10000$$

Final closed form: $P(n) = 10000 \cdot (1.06)^n$

Ex: $F(n) = F(n-1) + F(n-2)$, $F(0)=0$, $F(1)=1$

$x^2 - x - 1 = 0$, roots: $x_1 = \frac{1+\sqrt{5}}{2}$, $x_2 = \frac{1-\sqrt{5}}{2}$

So $F(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

Use base cases to find c_1 & c_2:

$F(0) = 0 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = c_1 + c_2$

$0 = c_1 + c_2$, or $c_1 = -c_2$

$F(1) = 1 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1$

$c_1 = -c_2$

$1 = (-c_2)\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$

$1 = \cancel{\frac{-c_2}{2}} - \frac{c_2\sqrt{5}}{2} + \cancel{\frac{c_2}{2}} - \frac{c_2\sqrt{5}}{2} = -c_2\sqrt{5}$

$\Rightarrow 1 = -c_2\sqrt{5} \Rightarrow c_2 = -\frac{1}{\sqrt{5}}$ $\quad c_1 = \frac{1}{\sqrt{5}}$

Ex: $B(n) = 2B(n-1) - B(n-2)$, $B(0)=0$, $B(1)=1$

$x^2 - 2x + 1 = 0$ had 1 root, $x = 1$, w/ multiplicity 2

so $B(n) = c_1 1^n + c_2 \cdot (n 1^n)$

Solve: $B(0) = 0 = c_1 \cdot 1^0 + c_2 \cdot 0 \cdot 1^0$

$$\Rightarrow c_1 = 0$$

$$B(1) = 0 \cdot 1^1 + c_2 \cdot 1 \cdot 1^1 = 1$$

$$c_2 = 1$$

So: $B(n) = \cancel{0 \cdot 1^n} + 1 \cdot n \cdot \underbrace{1^n}_{=1}$

$$B(n) = n$$

Ex: Spps we get char eqn for $C(n)$ as:

$$(x-2)^3(x-5)^2(x-9)=0$$

What is form of the general solution?

don't solve for constants

roots:

2 w/mult. 3

5 w/mult. 2

9 w/mult. 1

$$C(n) = c_1 9^n + c_2 5^n + c_3 n 5^n + c_4 2^n + c_5 \cdot n 2^n + c_6 \cdot n^2 \cdot 2^n$$

Linear

Dfn: Inhomogeneous recurrences have an added function $g(n)$:

$$f(n) = c_1 f(n-1) + \cdots + c_d f(n-d) + g(n)$$

Ex: $F(n) = F(n-1) + F(n-2) + 1$

$A(n) = 4A(n-1) + 3^n$

$+ n^2 + 2^n$

Method for inhomogeneous recurrences:

1. "Ignore" $g(n)$ and find general solution for the rest
   ↪ find char roots
2. Find general solution for $g(n)$ ← complex
3. Add them together
4. Use base cases (& possibly recurrence) to solve for constants.

We'll talk about how to do step 2 when $g(n) =$ (polynomial of degree k) $\cdot s^n$

(where s constant)

Ex: $g(n) = (n^2+1)\cdot 2^n$

$g(n) = (n+5)\cdot 1^n$

Is s a char root?

How to do it:

Is s a characteristic root?

No

try a general solution of the form (polynomial of degree k) $\cdot s^n$

use different constants

Yes

what is its multiplicity? Let this be m

try a general solution of the form n^m (poly. of degree k) $\cdot s^n$

Ex: $f(0) = 1$

$f(n) = 4f(n-1) + 3^n$ $\leftarrow$

① $f(n) = 4f(n-1)$

$x = 4$ root: $4 \Rightarrow c_1 \cdot 4^n$

$(x-4) = 0$

② $g(n) = 3^n = n^0 \cdot 3^n \Rightarrow s = 3$

$\Rightarrow (\text{poly deg } 0) \cdot 3^n = c_2 \cdot 3^n$

③ $f(n) = c_1 \cdot 4^n + c_2 \cdot 3^n$

④ $f(0) = 1 = c_1 \cdot 4^0 + c_2 \cdot 3^0 = c_1 + c_2$

$f(1) = 4 \cdot f(0) + 3^1 = 7 = c_1 \cdot 4^1 + c_2 \cdot 3^1$