Math 135 - Infinite Sets

Note Title 9/24/2010

Announcements

- Exam I is next Monday
- Review session is Friday in class
- HW due Friday

over everything up to last Friday (functions)

Ch 2.4: Sequences & Summations

Dfn: A sequence is a function from a subset of $\mathbb{Z}$ (usually $\mathbb{N}$) to a set S.

a_n = image of n under this function
= n^{th} term of the sequence

Ex: $a_n = \frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}$

Write $\left\{\frac{1}{n}\right\}_{n \in \mathbb{N}}$

Types of Sequences

Dfn: A geometric progression is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

Dfn: An arithmetic progression is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

We often consider summing such sequences:

$$a + ar + ar^2 + \cdots + ar^n = \sum_{i=0}^{n} a \cdot r^i$$

As we have already seen,

$$\sum_{i=0}^{n} a \cdot r^i = \begin{cases} \dfrac{a \cdot r^{n+1} - a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

(proved this with induction)

Double Summations:

Ex: $\displaystyle\sum_{i=1}^{4}\underbrace{\sum_{j=1}^{3} i j}_{} = \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i =$

$6(1+2+3+4)$

$= \boxed{60}$

$\displaystyle\sum_{j=1}^{3} i\cdot j = i + 2i + 3i = 6i$

$\displaystyle\sum_{i=1}^{4} 6i = (6\cdot 1 + 6\cdot 2 + 6\cdot 3 + 6\cdot 4) \Rightarrow 60$

Another one:

$$\underbrace{\sum_{i=1}^{n} \sum_{j=1}^{i} 1}_{} = \sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

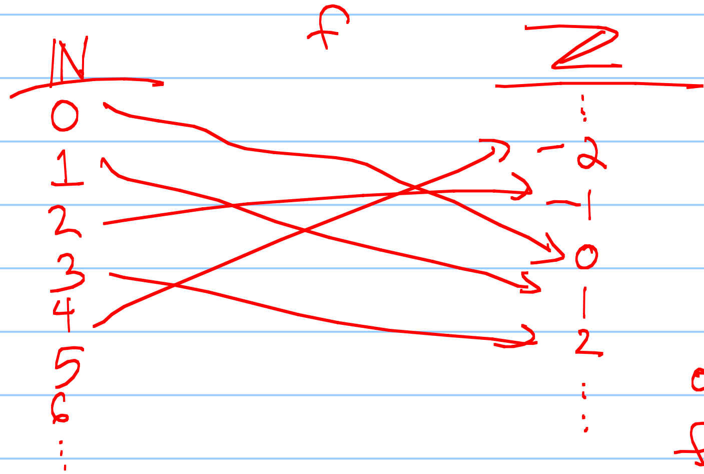
$$\sum_{j=1}^{i} 1 = \underbrace{1+1+\cdots+1}_{i \text{ times}} = i$$

Infinite Sets (Ch. 2.4, end of section)

Dfn: Two sets A + B have the same cardinality if and only if there is a bijection from A to B.

bijection = 1-1 and onto

Thm: $\mathbb{N}$ and $\mathbb{Z}$ have same cardinality.



$0 \longmapsto 0$

even $i \in \mathbb{N}$: $f(i) = -\frac{i}{2}$

$2 \mapsto -1$
$4 \mapsto -2$
$6 \mapsto -3$
$8 \mapsto -4$

odd $i \in \mathbb{N}$: $f(i) = \frac{i+1}{2}$

$1 \mapsto 1$
$3 \mapsto 2$
$5 \mapsto 3$
$7 \mapsto 4$
$9 \mapsto 5$

even i: $f(i) = -\frac{i}{2}$

odd i: $f(i) = \frac{i+1}{2}$

Claim: $\underbrace{\text{1-1}}$ and onto

$\forall a, b$, if $f(a)=f(b)$ then $a=b$

proof by contrapositive: suppose $a \neq b$

Case 1: both even: $f(a) = -\frac{a}{2}$ and $f(b) = -\frac{b}{2}$

Since $a \neq b$, we know $-\frac{a}{2} \neq -\frac{b}{2}$

so $f(a) \neq f(b)$

Case 2: both odd: $\frac{a+1}{2}$ $\frac{b+1}{2}$

Case 3: 1 even, + 1 odd: $f(a)$ would be positive + $f(b)$ negative.

$= \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$

Thm: $\mathbb{N}$ and $\mathbb{Q}^+$ have same cardinality.

sketch pf: need a bijection

trick: dovetailing

	-0	-1	-2	-3	-4	...
1	$-0/1$	$-1/1$	$-2/1$	$-3/1$		
2		$-1/2$				
3						
4						
5						
⋮						

$q \backslash p$	0	1	2	3	4	....
1	$0/1$	$1/1$	$2/1$	$3/1$	$4/1$	...
2	$0/2$	$1/2$	$2/2$	$3/2$	$4/2$	...
3	$0/3$	$1/3$	$2/3$	$3/3$	$4/3$	...
4	$0/4$	$1/4$	$2/4$			
5	$0/5$	$1/5$				
6						
⋮						

Are there sets which are "bigger" than $\mathbb{N}$?

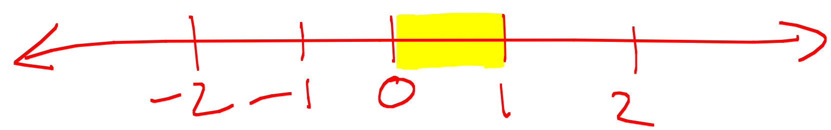
Dfn: A set A is countable if there is a bijection $f: \mathbb{N} \to A$ (or if A is finite).

So far: $\mathbb{Z}$ is countable

$\mathbb{Q}$ is countable

What about $\mathbb{R}$?

$\mathbb{R}$



Thm: $\mathbb{R}$ is not countable.

Actually, we'll show $(0,1) \subseteq \mathbb{R}$ is not countable.

Called Cantor diagonalization arguement:

$$.\,0\ 5\ 2\ 3\ 6\ \ldots$$

$$.\,d_1\ d_2\ d_3\ d_4\ d_5\ \ldots$$

pf: by contradiction

Suppose $(0,1)$ is countable.

$\exists\, f: \mathbb{N} \to (0,1)$ that is a bijection

That means we can list the numbers in $(0,1)$

$f(1) = .\,x_{1,1}\ x_{1,2}\ x_{1,3}\ x_{1,4} \ldots$

$f(2) = .\,x_{2,1}\ x_{2,2}\ x_{2,3}\ x_{2,4} \ldots$

$f(3) = .\,x_{3,1}\ x_{3,2}\ x_{3,3}\ x_{3,4} \ldots$

$f(4) = .\,0\ 1\ 7\ 2$

Show f can't be onto

Ex: $x = .\ x_{1,1}+1,\ x_{2,2}+1,\ x_{3,3}+1,\ 3$

In my example, i^{th} digit of x is not the same as i^{th} digit of $f(i)$
so $x \neq f(i)$ for any i. $\boxed{m}$