Math 135 - Proofs

Note Title 8/30/2010

Announcements

- HW1 due Friday
- Next HW will be posted by Friday, due on Monday the 13^{th}
  (will be harder!)
- HW0 is graded & will be returned on Friday

Proof techniques

What did we see last time?

- direct
- contradiction
- indirect
- proof by cases

$p \Rightarrow q$

$p \to q$

Thm: Suppose n is an integer. n is odd $\Longleftrightarrow$ n^2 is odd.

if and only if

pf: need to show "if n odd, then n^2 odd" and "if n^2 odd, then n odd"

① $\Rightarrow$: Assume n odd: $n = 2k+1$ for some $k \in \mathbb{Z}$.

$$n^2 = (2k+1)^2 = 4k^2+4k+1 = 2\underbrace{(2k^2+2k)}_{\text{integer}}+1$$

so n^2 is odd.

② $\Leftarrow$: Contrapositive: Assume n is even.

$n = 2\ell$ for $\ell \in \mathbb{Z}$

$n^2 = 4\ell^2 = 2(2\ell^2)$ so n^2 is even $\boxed{\text{:)}}$

Prove that if n is an integer, then $n^2 \geq n$.

① Suppose $n \geq 1$.
Multiply both sides by n:

$$n \cdot n \geq 1 \cdot n \Rightarrow n^2 \geq n.$$

② Suppose n is negative, so $n < 0$.
If n is negative, then n^2 is positive.
so $n^2 > 0$.

$$n < 0 < n^2 \Rightarrow n < n^2$$

③ $n = 0$, then $n^2 = 0$, so $0 \geq 0$. $\boxtimes$

Show that there exist irrational numbers x and y such that x^y is rational.

pf: We know $\sqrt{2}$ is irrational.

Consider $\sqrt{2}^{\sqrt{2}}$.

If $\sqrt{2}^{\sqrt{2}}$ is rational, then let $x = \sqrt{2}$, $y = \sqrt{2}$.

If $\sqrt{2}^{\sqrt{2}}$ is irrational, let $x = \sqrt{2}^{\sqrt{2}}$, & let $y = \sqrt{2}$

Then $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2}$

$= 2.$

KB

Induction

A proof technique that is used to prove propositions of the form:

$$\underbrace{\forall n}, \; P(n)$$

$\forall n \geq 1, \; \forall n > 4, \; \forall n \geq c$

Idea:

1. Show $P(1)$ true
2. Show $\forall k > 1, \; P(k-1) \rightarrow P(k)$

Since $P(1)$ is true (by 1):

$P(1) \rightarrow P(2)$ (by 2)

$P(2) \rightarrow P(3)$ (by 2)

$P(3) \rightarrow P(4)$ (by 3)

$\vdots$

Example: $\forall n \geq 1, \overbrace{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}^{P(n)}$

proof: by induction on n

① Show $P(1)$ is true

need to show: $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$ $\qquad$ $\sum_{i=1}^{1} i = 1$ & $\frac{1(2)}{2} = 1$

so $P(1)$ holds

② $\underbrace{\forall k > 1, P(k-1) \to P(k)}$:

Assume $\sum_{i=1}^{k-1} i = \frac{(k-1)(k)}{2}$. Goal is to show $P(k)$.

$\sim 1+2+\cdots+(k-1)$

$$\sum_{i=1}^{k} i = \underbrace{1+2+3+\cdots+(k-1)}+k = \sum_{i=1}^{k-1} i + k = \frac{(k-1)k}{2} + k$$

$$= k\left(\frac{k-1}{2}+1\right) = k\left(\frac{k-1+2}{2}\right)$$

How to write inductive proofs

3 required parts

Base Case: Show $P(1)$ is true

Inductive Hypothesis: Assume $P(k-1)$

Inductive Step: Use IH to argue that $P(k)$ is true.

Example: Show that the sum of the first n odd integers is n^2.

$$\sum_{i=1}^{n} (2i-1) = n^2$$

$$(1+3+5+\dots+\underbrace{2n-1}) = n^2$$

proof: Base case: $\sum_{i=1}^{1} (2i-1) = 1$ ✓

$$n^2 = 1^2 = 1$$

Ind. Hyp.: Assume $\sum_{i=1}^{n-1} (2i-1) = (n-1)^2$

Ind. Step: $$\sum_{i=1}^{n}(2i-1) = \underbrace{\sum_{i=1}^{n-1}(2i-1)} + (2n-1)$$

$$= (n-1)^2 + (2n-1)$$

(by Ind. Hyp.)

$$= n^2 - 2n + 1 + (2n-1)$$

$$= n^2$$