Math 135 - Functions

Note Title 9/17/2010

Announcements

- HW due Friday
- Next HW due next Friday
- Midterm on Monday the 4th
- Office hours today 1-2
  tomorrow 11:30 - 12:30
  (9-10)

A note about homework:

"Prove or disprove".

Prove — requires a proof for full credit!

Disprove — requires a counterexample

(or a proof that it does not hold, but usually counterexample is easier)

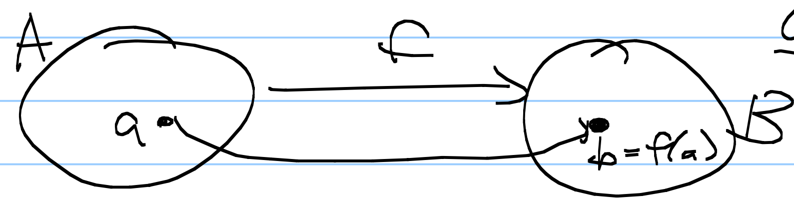
Functions

Let A & B be sets. A <u>function</u> from A to B is an assignment of exactly one element of B to each element of A.

We write $f(a) = b$ where $a \in A$, $b \in B$.

Often write $f: A \to B$ to denote a function f.

A is the <u>domain</u> of f, & B is the <u>co-domain</u>.



Examples

1. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x+1$

2. Truth table $\{T,F\} \times \{T,F\} \to \{T,F\}$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

3. Let $X = \{a,b,c\}$ and $c: P(X) \to P(X)$ be the function:

2^X = set of all subsets of X

$$c(A) = X - A$$

Dfn: A function $f: A \to B$ is <u>one-to-one (1-1), or injective,</u> if & only if $f(a) = f(b)$ implies $a = b$.

Such a function is said to be an <u>injection</u>.

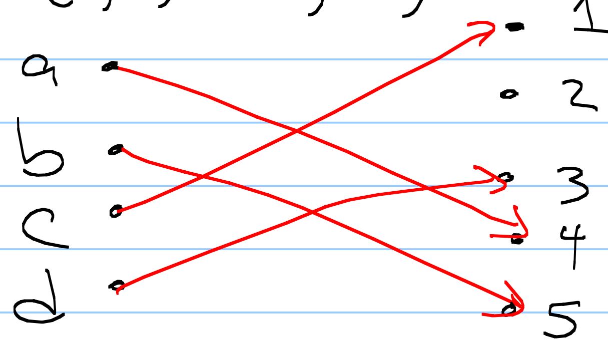
logic notation:

$$\underbrace{\forall a, b \in A,\ f(a) = f(b) \Rightarrow a = b}$$

So for these functions, no element in B has more than one element of A mapping to it.

Ex: $f: \{a, b, c, d\} \to \{1, 2, 3, 4, 5\}$ Yes

$f(a) = 4$
$f(b) = 5$
$f(c) = 1$
$f(d) = 3$



Ex: $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$

is it injective? No

$$f(a) = (-2)^2 = 4 = 2^2 = f(b)$$

but $a = -2 \neq 2 = b$

Prove that $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x+1$, is injective.

<u>Pf:</u> Need to show that $f(x) = f(y) \to x = y$.

Suppose $f(x) = f(y)$

$$x + 1 = y + 1$$

subtract 1 from each side

$$\Rightarrow \quad x = y$$

$\blacksquare$

Dfn: A function is called onto (or surjective) if and only if for every element $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

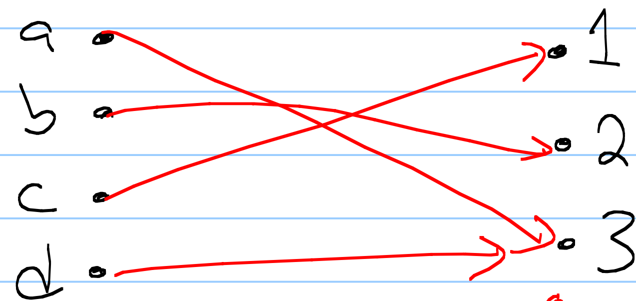
In logic:

$$\underbrace{\forall b \in B \ \exists a \in A \ \text{s.t.} \ f(a) = b}$$

So for these functions, every element of B must be an "output" of f.

Examples (1) $f: \{a, b, c, d\} \to \{1, 2, 3\}$

$f(a) = 3$
$f(b) = 2$
$f(c) = 1$
$f(d) = 3$



everything gets "hit"

Onto? Yes

1-1? No: $f(a) = f(d)$ but $d \neq a$

(2) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$

Is it onto? No

Ex: $-3, 3, 5$

(lot of examples)

(3) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = x+1$

onto?

take any $n \in \mathbb{Z}$

If we subtract 1, know $n-1 \in \mathbb{Z}$

and $f(n-1) = n$

$\square$

Dfn: A function is a bijection if it is both 1-1 and onto.

Ex: $f(x) = x+1, \quad f: \mathbb{Z} \to \mathbb{Z}$

Dfn: The *identity function* on A, $i_A : A \to A$, is the function $\underbrace{i_A(a) = a}$ $\forall a \in A$.

Ex: $i_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$

$i_{\mathbb{N}}(x) = x$

$A = \{1, 2, 3\}$

$i_A(1) = 1$

$i_A(2) = 2$

$i_A(3) = 3$

Dfn: Suppose f is a bijection. The inverse of f, written f^{-1}, is the function

$$f^{-1}: B \to A \quad \text{where} \quad \underbrace{f^{-1}(b) = a \iff f(a) = b}.$$

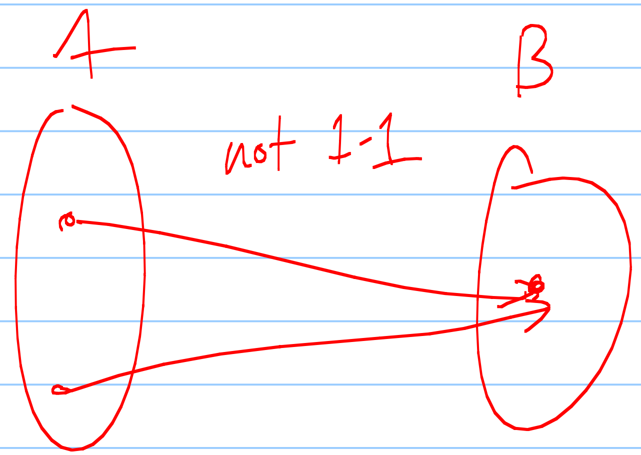
Ex: What is the inverse of $f: \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x+1$?

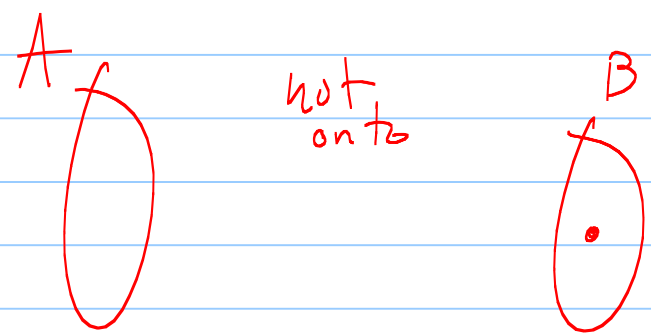
$f(2) = 3$

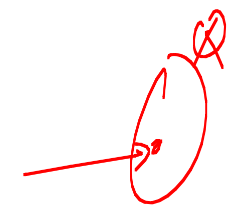
$f^{-1}(3) = 2$

$$f^{-1}(y) = y - 1$$

Only *bijective* functions have inverses.







Ex: $f: \mathbb{Q} \to \mathbb{Q}$, $f(x) = \frac{x}{2} + 3$

is it 1-1? yes

Suppose $f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right)$

$$\frac{(a/b)}{\cancel{2}} \cancel{+3} = \frac{(c/d)}{\cancel{2}} \cancel{+3}$$

is it onto? take $\frac{p}{q} \in \mathbb{Q}$

Find what maps to p/q

$$x = \underbrace{2\left(\frac{p}{q} - 3\right)} \in \mathbb{Q}$$

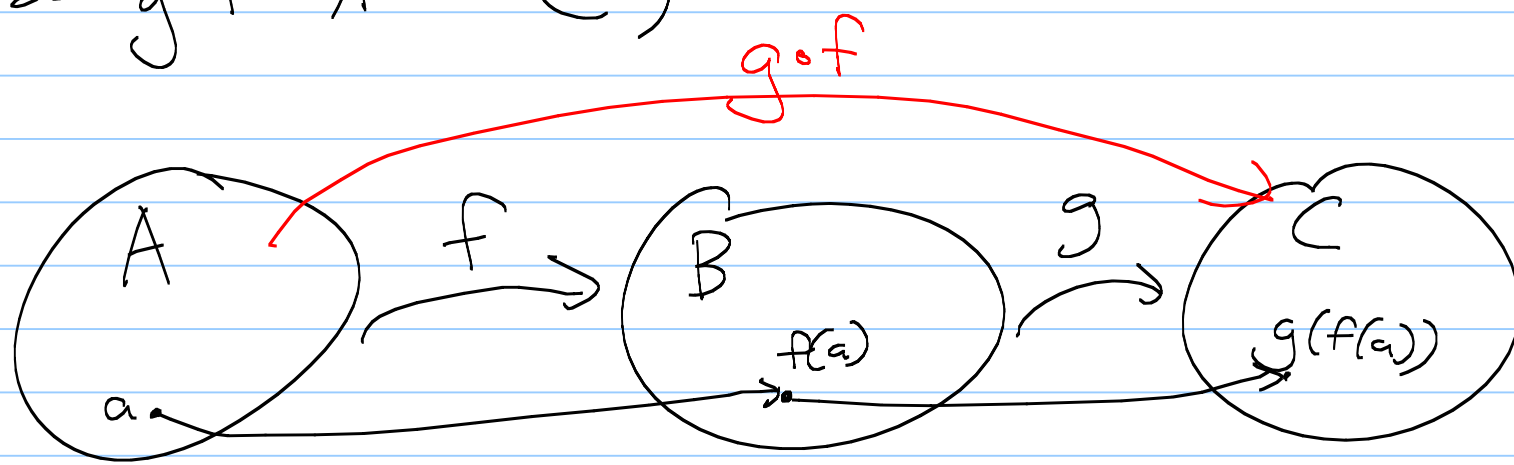
and $f(x) = \frac{p}{q}$

Composition of functions

Given $f: A \to B$ and $g: B \to C$, the <u>composition</u> of f and g, written $g \circ f$, is the function

$$(g \circ f)(a) = g(f(a))$$

$\left(\text{so } g \circ f : A \longrightarrow C \right)$



Ex: Let $f: \mathbb{Z} \to \mathbb{Z}$ with $f(x) = 2x+3$
and $g: \mathbb{Z} \to \mathbb{Z}$ with $g(x) = 3x+2$.

What is $g \circ f$?

$g \circ f: \mathbb{Z} \to \mathbb{Z}$

$$(g \circ f)(x) = g(f(x)) = g(2x+3)$$
$$= 3(2x+3) + 2$$
$$= 6x + 11$$

Thm: Functions $f: A \to B$ and $g: B \to A$ are inverses of each other if and only if $f \circ g = i_B$ and $g \circ f = i_A$.

Proof: "$\Longrightarrow$": Assume $f: A \to B$ and $g: B \to A$ are inverses of each other.

by definition

Know $g(b) = a \iff f(a) = b$.

Take any $a \in A$. Consider

$$(g \circ f)(a) = g(f(a)) = g(b) = a$$

So for any a, $(g \circ f)(a) = a = i_A(a)$. $\square$

proof cont: "$\Leftarrow$": Suppose $\underbrace{g \circ f = i_A}$ & $\underbrace{f \circ g = i_B}$

$\underbrace{\text{Show } f \text{ \& } g \text{ are inverses.}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Vert$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f(a) = b \Leftrightarrow g(b) = a$

Take $a \in A$, $f(a) = b$,

Consider $(g \circ f)(a) = g(f(a)) = g(b)$

$\qquad\qquad\quad \Vert$

$\qquad\qquad i_A(a) = a$

So far, if $f(a) = b \Rightarrow a = g(b)$

Take $y \in B$, let $g(y) = x$.

$\underbrace{(f \circ g)}_{i_B}(y) = f(g(\cancel{y})) = f(x)$

$\qquad\qquad\quad \Vert$

$\qquad\qquad\quad y$

$\boxed{\text{Ш}}$

Next time:

Finish functions

summations + sequences