Math 135 - Counting (part 3)

Note Title 11/18/2010

Announcements

- Next HW is posted due ~~Monday~~ Tuesday before 5pm
  bring it to my office (slide under door)

- Final exam is 12-2 on first day of finals (Wed.)

Permutations

How many ways are there to list r distinct elements from a set of size n?

Dfn: Call this $P(n,r)$. $= \dfrac{n!}{r!}$

rule of product: $\underline{n}\ \underline{n-1}\ \underline{n-2}\ \ldots\ \underline{n-r+1}$

Formula: $P(n,r) = \dfrac{n!}{(n-r)!}$

$= n(n-1)(n-2)\cdots(n-r+1)$

Ex: Suppose we have 8 runners, & will award 3 medals (gold, silver & bronze). Assuming no ties, how many different possible ways to award?

$$\frac{8!}{5!} = 8 \cdot 7 \cdot 6$$

Ex: How many permutations of the alphabet contain the string 'ABC'?

$\underbrace{\text{'ABC'}}_{\text{one letter}}$

$P(24,24) = 24!$

23 other letters

+ "ABC"

Combinations

order doesn't matter!

How many ways are there to choose r elements out of n?

Notation: $C(n,r) = \binom{n}{r} =$ "n choose r"

book

everyone else

what we say

Ex: How many ways to choose 2 elements from $\{1, 2, 3, 4, 5\}$?

Ans: $\{1,2\}$ $\{1,3\}$ $\{1,4\}$ $\{1,5\}$

$\{2,3\}$ $\{2,4\}$ $\{2,5\}$

$\{3,4\}$ $\{3,5\}$

$\{4,5\}$

$$\binom{5}{2}$$

Thm: $P(n,r) = \underbrace{\binom{n}{r}}_{\substack{\text{\# of ways to} \\ \text{choose } r \text{ people}}} \cdot \underbrace{P(r,r)}_{\substack{\text{\# ways to order} \\ \text{those } r \text{ people}}}$

pf:

This gives a nice formula:

$$\binom{n}{r} = \frac{P(n,r)}{P(r,r)}$$

$$= \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!\,r!}$$

Ex: How many possible poker hands are there?

(52 different cards in deck, 5 card hands)

$$\binom{52}{5} = \frac{52!}{5!\,47!}$$

Ex: How many bit strings of length 5 have exactly 3 ones?

$\binom{5}{3}$

$\underline{\ \ }\ \underline{1}\ \underline{\ \ }\ \underline{1}\ \underline{1}$

Choose 3 spots in which to place 1s

(Follow-up: How many bitstrings of length n have exactly r ones?)

$\binom{n}{r}$

Combinatorial Proof:

a proof which uses counting arguements to prove that both sides of an identity count the same thing

Ex: $\underbrace{\binom{n}{r}}_{\text{\# of bit strings with } r\ 1\text{'s}} = \underbrace{\binom{n}{n-r}}_{\text{pick spots for } n-r\ 0\text{'s}}$

Thm: $$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$$

pf: by combinatorial argument

LHS: $n+1$ people & we are choosing a committee of size $r+1$

RHS: Isolate 1 person - either on committee or not. If on, choose r others. If not, choose $r+1$.

Use rule of sum, since these possibilities are independent.

Thm (Vandermonde's identity)

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

in text

Permutations with repetition

How many strings of length r can be formed from English alphabet?

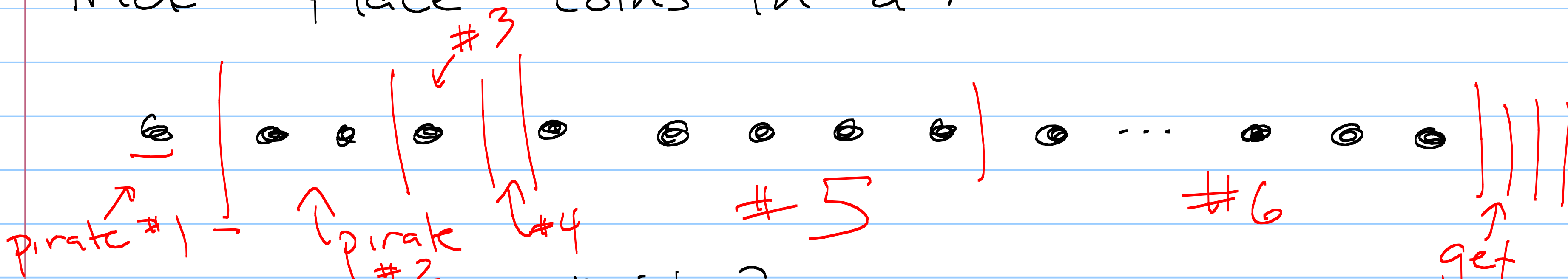
$$\underbrace{\underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} \cdots \underline{26}}_{r \text{ spots}} = 26^r$$

[Note - not $P(26, r) = 26 \cdot 25 \cdot 24 \cdots (26-r+1)$]

Combinations

How many ways are there to distribute r identical gold coins among n pirates?

Trick: Place coins in a row:



How can we divide?

n piles (one per pirate)

In total, have $\underbrace{r}_{\#\text{ coins}} + \underbrace{n-1}_{\#\text{ bars}}$

Need to choose r spaces for the coins; rest become bars

$$\binom{r+n-1}{r} = \text{\# ways to give } r \text{ coins to } n \text{ pirates}$$

(substitute 0's & 1's)

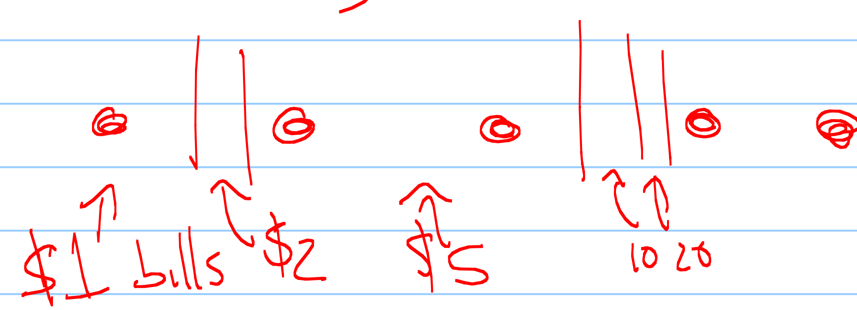
Q: How many ways are there to select 5 bills from a cash drawer containing \$1 bills, \$2 bills, \$5, \$10, \$20, \$50, and \$100 bills?

(Assume bills of same type are indistinguishable, and that we have at least 5 of each type.)

7 kinds of bills $= n$ (pirates)

5 bills (coins)

$$\binom{5+7-1}{5}$$



Q: Suppose a cookie shop has 4 different kinds of cookies. How many different ways to choose 6 cookies?

4 types = pirates n

6 coins / cookies = r

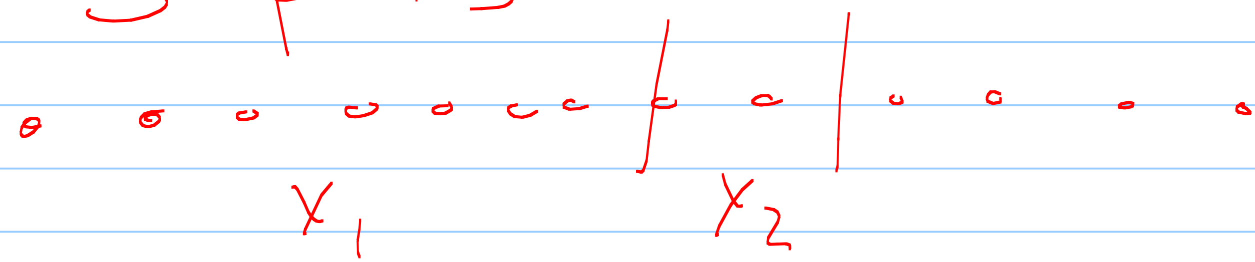
$$\binom{6+4-1}{6} = \boxed{\binom{9}{6}}$$

Q: How many non-negative integer solutions are there to:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100?$$

100 coins

5 pirates



$$\binom{100+5-1}{5} = \binom{104}{5}$$