Note Title 4/7/2010

Math 135 - More Counting (5.1, 5.2)

Announcements

- HW due Monday
- Review on Monday
- Exam on Wednesday

Last time:

1. Rule of Sum – when 2 sets are disjoint
2. Rule of Product – series of choices
3. Inclusion-Exclusion: when 2 sets have an $\cap$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Sec. 5.2 : The Pigeonhole Principle

Thm: If k is a positive integer and $k+1$ balls are placed into k boxes, then some box contains 2 or more balls.

Proof: Contrapositive - say no box contains 2 balls.

Then we have at most k balls.

Examples:

- A function from a set with $k+1$ elements to a set with k elements is not 1-1.

  boxes = elements in codomain (k)
  balls = elements in domain $(k+1)$

- In any group of 367 people, 2 have the same birthday.

  boxes = days of year
  balls = people

  (birthdays put "balls" into "boxes")

- In any group of 27 words, some 2 start with the same letter.

  boxes = alphabet
  balls = words

Better: Show that for every integer n, there is a multiple of n that is written with only 0's & 1's (in decimal).

boxes = remainder when a number is divided by n

→ n boxes

$n+1$ numbers:

$$\left.\begin{array}{l} 1 \\ 11 \\ 111 \\ 1111 \\ \vdots \\ \underbrace{1 \ldots\ldots 1}_{n+1 \text{ 1's}} \end{array}\right\} \text{balls}$$

Some 2 #s from my list have same remainder when divided by n, by pigeon hole p.

$\underbrace{11\ldots1}_{k}$ $\quad$ $\underbrace{1\ldots1}_{\ell}$

Subtract smaller from larger.

$n = 10$

1
11
111
1111
⋮

$111111 - 1 =$

11 $\qquad 1111 - 11 = 1100$

$x \cdot y$

For any 2 numbers in same remainder class, $x - y$ is divisible by n.

$n=8$

1 – rem 1
11 – rem. 3
111 rem 7
1111 – rem. 7
11111 – rem. 7
111111
1111111
11111111
111111111

$1111 - 111 = 1000$ div. by 8

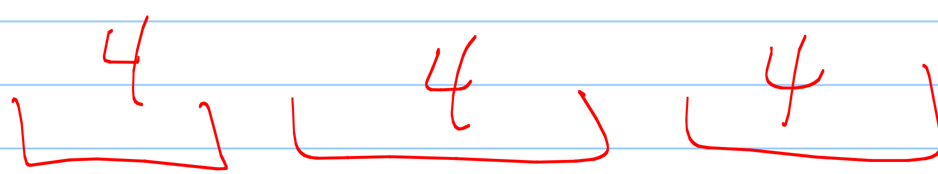
$\frac{1000}{8} = 125$

Generalized pigeonhole principle

If N balls are placed into k boxes, then there is a box containing at least $\lceil N/k \rceil$ balls.

~~pf:~~ Ex: 13 balls
3 boxes

some box will have $\geq \lceil \frac{13}{3} \rceil = 5$ balls in it.



Example: Among 100 people, how many must be born in same month?

boxes = months, so $k = 12$

balls = people, so $N = 100$

$$\left\lceil \frac{N}{k} \right\rceil = \left\lceil \frac{100}{12} \right\rceil = \boxed{9}$$

Ex: How many cards must we select from a standard 52 card deck in order to be sure that 3 are of the same suite?

boxes = suites so $k=4$

balls = cards that we pick

$N = ?$

$$\left\lceil \frac{N}{4} \right\rceil = 3$$

$N=9$

$\frac{1}{1}$ $\frac{2}{3}$ $\frac{3}{2}$ $\frac{4}{1}$

$a_1 = 1$ $a_2 = 4$ $a_3 = 6$ $a_4 = 7$

Ex: During a month with 30 days, a baseball team plays at least 1 game a day, but no more than 45 total.
Show that there is a period of consecutive days where the team plays exactly 14 games.

Let a_i = number of games played on or before the ith day of the month.
$1 \le a_1, \ldots, a_{30} \le 45$

Create another list, $15 \le a_1 + 14, a_2 + 14, \ldots, a_{30} + 14 \le 59$

Consider all 60 of these numbers. ← balls
boxes are #s $1 \ldots 59$
So some $a_i = a_k + 14$ $\boxed{}$

$$\frac{2^k}{2^\ell} = 2^{k-\ell}$$

$a_1 \ldots a_{n+1}$

__Ex:__ Among any $n+1$ positive integers which are $\le 2n$, one integer must divide another.

For any number a_i, rewrite as

$$a_i = 2^{k_i} q_i$$

$22 = 2^1 \cdot 11$
$16 = 2^4 \cdot 1$
$13 = 2^0 \cdot 13$

Balls: $a_1 \ldots a_{n+1}$

__boxes:__ odd numbers b/t $1 \ldots 2n$

↳ how many? n

Know: Some a_i & a_j had same q

$a_i = 2^{k_i} q$ and $a_j = 2^{k_j} \cdot q$

So one divides the other.